CATEGORY: Genetic Programing

Individual GP: an Alternative Viewpoint for the Resolution of Complex Problems.

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Abstract

An unususal GP implementation is proposed, based on a more "economic" exploitation of the GP algorithm: the "individual" approach, where each individual of the population embodies a single function rather than a set of functions. The final solution is then a set of individuals. Examples are presented where results are obtained more rapidly than with the conventional approach, where all individuals of the final generation but one are discarded.

1 Introduction

We present a more "economic" approach of the resolution of some complex problems such as the ones related to Iterated Function Systems: it is based on the co-evolving capacities of populations in GA/GP. The solution to the problem is then represented by the whole population (or a subset of the whole population) and not any more by a single individual (just as "classifier systems" approaches, see [10], or as in [14] for Evolution Strategy). We have chosen to call this approach "individual." Although it is more complex to implement (mainly with respect to the fitness computations) it allows to build more efficient algorithms in some particular cases.

We describe the general characteristics of such an approach in section 2. We then present how it can be applied in an efficient way to problems related to the study of some fractal objects (used for image compression); 2D attractors of non-linear Iterated Function Systems (IFS). Theoretical background for IFS and Polar IFS is presented in section 3. Section 4 presents an application to the random generation of 2D Polar-IFS attractors with a fixed surface. Section 5 describes how individual GP has been used to solve the inverse problem for Polar IFS.

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2 Individual GP

The standard approach, which uses evolutionary methods as stochastic optimisers (where a set of individuals in the search space evolves, via specific, classical or genetic operators, so that the best individual of the population converges towards the desired optimum) may sometimes seem wasteful: only the best individual of the final population is kept, while the others are discarded. The behaviour of GA however leads us to think that an important part of the final population bears significant information on the structure of the search space. This observation has led to, and justified such techniques as *sharing*, or *niching* (see [10]) that get more out of evolutionary algorithms than only guiding the best individual towards the global optimum.

If the solution to the problem is represented by an important set of individuals, or by the whole population[1], the implementation of the algorithm is more delicate:

- Not all optimisation problems can be formulated as a union of sub-problems.
- One must be able to correctly evaluate the contribution of each of the individuals to the global solution (one can quite often use a local evalua-

[1] This approach is not new, and is usually referred to as "Michigan" type GA's

tion function for an individual along with a global evaluation function for the whole population).

- Finally, it seems indispensable to use a "sharing" method with a well chosen distance function to place each individual with reference to the others. The evolution of the system can be seen as seeking a position of balance.

3 Fractal shapes based on Iterated Function Systems

3.1 IFS Theory

An IFS (Iterated Function System) $\mho = \{E, (w_n)_{n=1,..,N}\}$ is a collection of N functions defined on a complete metric space (E, d).

Let W be the operator defined on the space of subsets of E^2:

$$\forall\ K\ \subset\ E,\ \ W(K) = \bigcup_{n \in 1,\ldots,N} w_n(K)$$

Then, if the w_n functions are contractive (the IFS is then called a *hyperbolic* or *contractive* IFS), there exists a unique set A such that: $W(A) = A$. A is called the **attractor** of the IFS.

> *Recall:*
> A mapping $w : E \to E$, from a metric space (E, d) into itself, is called **contractive** if there exists a positive real number $s < 1$ such that:
>
> $$d\big(w(x), w(y)\big) \leq s \cdot d(x, y) \quad \forall x, y \in E$$

The uniqueness of a hyperbolic attractor is a result of the Contractive Mapping Fixed Point Theorem for W, which is contractive according to the HAUSDORFF distance:

$$d_H(A, B) = \max\Big(\max_{x\in A}\big(\min_{y\in B} d(x, y)\big), \max_{y\in B}\big(\min_{x\in A} d(x, y)\big)\Big)$$

From a computational viewpoint, an attractor can be generated according to two techniques:

- **Stochastic method (toss-coin):**
  Let x_0 be the fixed point of one of the w_i functions. We build the points sequence x_n as follows: $x_{n+1} = w_i(x_n)$, i being randomly chosen in $\{1..N\}$.
  Then $\bigcup_n x_n$ is an approximation of the real attractor of $\mho$. The larger n, the more precise the approximation.

- **Deterministic method:**
  From any initial set S_0, we build the sets sequence $\{S_n\}$:

  $$S_{n+1} = W(S_n) = \bigcup_n w_n(S_n)$$

  When n tends towards ∞, S_n is an approximation of the real attractor of $\mho$.

[2] $w_n(K)$ represents the set $\{w_n(x), x \in K\}$

3.2 Polar IFS

Problems associated to affine IFS, i.e.: when the w_i are affine 1D or 2D functions, have been extensively studied, mainly because fractal compression techniques rely on affine IFS modelling. A major challenge is to tackle the inverse problem for non-affine IFS. Previous work on this subject have raised the idea to use GP for the resolution of such problems, [15], [5].

The main problem which arises when manipulating non-linear IFS (mixed IFS, [15], for instance) is the management of the contractance constraint. This is quite tricky when one tries to solve the associated inverse problem using stochastic methods.

Let us use a subset of non-linear functions, w_i, conctracting with respect to a point P_i:

$$\forall M \in E = [0,1]^2 \qquad ||\overrightarrow{P_i w_i(M)}|| < ||\overrightarrow{P_i M}|| \qquad (1)$$

which can be transcribed in polar coordinates centred on P_i as:

$$\overrightarrow{P_i w_i(M)} = \begin{pmatrix} \rho \dfrac{th(k * F(\rho,\theta)) + 1}{2} \\ G(\rho,\theta) \end{pmatrix} \qquad (2)$$

$F(\rho,\theta)$ and $G(\rho,\theta)$ are random non-linear functions which can be represented with a tree (as for mixed-IFS functions).

The form $\rho\frac{th(k*F(\rho,\theta))+1}{2}$ insures that the relation (1) is verified, because the factor $\frac{th(k*F(\rho,\theta))+1}{2}$ is always < 1. The form of this factor has been chosen in order to make a rather smooth bijective mapping of $\mathbb{R}$ onto $(0,1)$, see figure 1. k is fixed to 10^{-7} for the same reasons.

The fixed points P_i of these w_i functions are:

$$\forall M \in E \quad \lim_{n\to\infty} w_i^n(M) = P_i$$

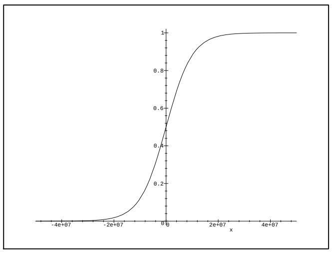


Figure 1: The $y = \frac{th(kx)+1}{2}$ curve with $k = 10^{-7}$

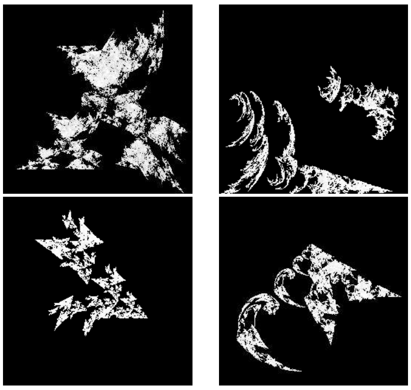


Figure 2: Examples of Polar IFS attractors

However, these functions are not systematically contractant, see [5] for details.

The restriction to functions verifying (2) does not unfortunately prevent us from checking whether the functions are contractant or not. This restriction is however very interesting, because functions constructed this way are quite often contractant and have a known unique fixed point. What is more, this set of functions is general enough to produce a wide variety of shapes through their attractor, see figure 2. They are quite easy to use in such applications as the solving of the inverse problem or the interactive generation of attractors.

4 Random generation of Polar IFS attractors with a prescribed surface

A first problem that can be solved using GP is the random generation of non-linear IFS attractors. Due to the contractance constraints, the space of possible sets of w_i functions which admit an attractor is very sparse. If one wants to find IFS attractors that have compact, "nice," or fixed criteria, the problem becomes extremely difficult. Solutions have been proposed in [18] in an interactive manner for the "artistic" generation of attractors, similar to Karl Sims techniques [19].

This approach is based on a conventional exploitation of GP: a whole IFS (i.e.: a set of w_i functions) is represented as an individual of a population.

If an individual of the population encodes a single w_i function, the IFS is represented by the whole population, or by a part of the population. We present below how this structure can be evolved to produce an IFS having an attractor of arbitrary surface in the image.

The advantage of using Polar IFS is twofold:

1. each function converges towards a fixed point, and the functions are rather frequently contractive, see [5] for experiments,

2. access to the fixed point of the function is direct.

This second point allows to consider in a different way genetic operators on fixed points and on tree structures of an individual that represents a w_i function. Genetic operators are classical GP *mutation* and *crossover* for the tree structured part of the w_i. Fixed points are mutated according to a random shot in a disk of radius σ centered on it. Fixed points crossover is a classical barycentric crossover whose parameter is randomly chosen in $[-1, 2]$ for each offspring ($x' = \gamma x + (1-\gamma)y$).

The main problem of "individual" approaches is the careful design of the fitness function, and the use of a sharing scheme, in order to disperse the individuals of the population.

The fitness function can be made up of two main contributions: a local fitness which measures the intrinsic characteristic of the individual, and a global fitness that stems from the global performance of the population, redistributed on each individual, proportionally to its "contribution."

- **Local Fitness:** a combination of three terms according to:

  1. *The fixed point position* with respect to the image (represented as $\Box$, the $[0, 1]^2$ square). A very simple property of attractors is that the fixed point of each w_i belongs to the attractor of $\{w_i\}$. If we thus wish that the attractor of the $\{w_i\}$ be inside the image, the fixed points have to be within $\Box$.
     A first term is proportional to the inverse distance between the fixed point x_i of w_i and $\Box$:

     $$F_1(w_i) = \frac{1}{1 + D(x_i, \Box)}$$

     F_1 is maximum and equals 1 when $x_i \in \Box$, and tends to zero when x_i is far away from $\Box$.

Table 1: Parameters setting for the random generation of attractors, using GA-Lib

SIGMA	0.2
Local fitness tuning	$\alpha = 20$
Mutation probabilities	
constant $\rightarrow$ constant	0.15 according to a Gaussian law of variance SIGMA
variable $\rightarrow$ constant	0.02 randomly chosen in $[-1, 1]$
constant $\rightarrow$ variable	0.06
variable $\rightarrow$ variable	0.08
function $\rightarrow$ function (same arity)	0.08
fixed points:	0.03 according to a uniform law in the circle of radius SIGMA
Crossover probability	
PCROSS	0.95 for trees and fixed points
Sharing	
σ (Goldberg [10])	2*SIGMA
Population replacement scheme	
replacement percentage	50% Overlaping populations

2. *The $w_i(\Box)$ position with respect to $\Box$:* We compute the image $w_i(\Box)$, in order to test if it is included in $\Box$.

   Let $\#[X]$ be the number of pixels (according to the considered image resolution) of the set X, then:

$$F_2(w_i) = \frac{\#[w_i(\Box) \bigcap \Box]}{\#[w_i(\Box)]}$$

   $F_2(w_i)$ is maximum (and $= 1$) if $w_i(\Box) \subset \Box$.

3. *The size of $w_i(\Box)$ in $\Box$:*

$$F_3(w_i) = \frac{\#[w_i(\Box) \bigcap \Box]}{\#[\Box]}$$

   We have chosen to favour w_i's that generate large images.

The local fitness for individual w_i is:

$$F_{loc}(w_i) = F_1(w_i) + F_2(w_i) + \alpha F_3(w_i) \qquad (3)$$

α tunes the relative importance of term F_3 in comparison to F_1 and F_2 (F_1 and F_2 tend easily to one, while F_3 is more difficult to increase).

- **Contractivity constraints:**
  The contractance test can be included in the computation of the image of $w_i(\Box)$. At the same time, the mean contraction factor k_i can be estimated. If the function is not contractive, F_2 is not computed and is directly fixed to zero, as well as F_{loc} in order to discard this individual.

- **Global fitness:**
  The N (to be determined with respect to the local fitness[3]) best individuals of the evolved population represent a solution to our problem. A toss-coin algorithm can thus be used in order to compute the attractor $\mho$ of these individuals, and a global fitness can be defined for a prescribed image occupancy $S \in [0, 1]$ as:

$$F_{glob} = \frac{2}{1 + 100(\frac{\#[\mho]}{\#[\Box]} - S)^2}$$

  F_{glob} is a measurement of the distance between $\frac{\#[\mho]}{\#[\Box]}$ and S. The function has been chosen so that $F_{glob} = 1$ when $\#[\mho] = S \pm 10\%$.

  This global fitness can be distributed on the N w_i which have been selected from the current population (the global fitness of the individuals that have not been selected is simply F_{loc}), proportionnally to their contribution to $\mho$ i.e.: to $F_2(w_i)$, or grossly to k_i (in fact related to $\overline{k_i} = \frac{\sum k_j}{N}$):

$$F(w_i) = F_{loc}(w_i) \times N \frac{k_i}{\sum k_j} F_{glob} \qquad (4)$$

  F_{glob} is used as a multiplicative factor, thus improving (if ≥ 1) or degrading the individuals' fitness with respect to their global performance.

  F_{glob} can also be used as a stopping criterion for the GP: stop the algorithm when the target surface is approximated with a fixed threshold.

A GP with sharing is used, the distance being simply the euclidean distance between fixed points of the w_i functions.

Results obtained with the parameter setting of table 1 are presented in figures 3 and 4.

5 Resolution of the inverse problem for Polar IFS

The inverse problem for 2D IFS can be stated as follows:

[3] In fact, we select all the contractive individuals of the population with $F_3(w_i) > 0.1$

density 0.24, 10 generations, 6 functions | density 0.2, 9 generations, 7 functions | density 0.22, 20 generations, 10 functions

Figure 3: Three different runs of the algorithm: 128x128 random attractors generated for $S = 0.2$, with a population size of 20 individuals, the algorithm is stopped as soon as $\frac{\#[\mho]}{\#[\square]} > S$.

density 0.41, 53 generations, 12 functions | density 0.41, 54 generations, 10 functions | density 0.41, 91 generations, 19 functions

Figure 4: Three different runs of the algorithm: 128x128 random attractors generated for $S = 0.4$, with a population size of 30 individuals, the algorithm is stopped as soon as $\frac{\#[\mho]}{\#[\square]} > S$.

> *for a given 2D shape (a binary image), find a set of contractive maps whose attractor produces a similar shape, the similarity being determined by a pre-defined distance function.*

An interesting tool for the resolution of the inverse problem is the so-called collage theorem [2]:

Collage theorem: Let A be the attractor of the hyperbolic IFS $\mho = \{E, (w_n)_{n=1,..,N}\}$:

$$\forall K \subset E, d_H\big(K, W(K)\big) < \varepsilon \quad \Rightarrow \quad d_H(K, A) < \frac{\varepsilon}{1-\lambda}$$

λ being the smallest number such that:

$$\forall n \in \{1,..,N\}, \forall (x,y) \in E^2,$$
$$d\big(w_n(x), w_n(y)\big) < \lambda \cdot d(x,y)$$

This theorem means that the problem of finding an IFS $\mho$ whose attractor is close to a given shape A, is equivalent to the minimisation of the distance:

$$d_H\big(A, \bigcup_{i=1}^{n} w_i(A)\big)$$

under the constraint that the w_i are contractive functions.

We will see below that the "individual" approach allows to use information stemming from both collage theorem and toss-coin algorithm, in order to solve the inverse problem efficiently.

In the same way as in section 4, each w_i is mainly evaluated as a function of the position of its fixed point (which is always defined and known, thanks to the use of polar IFS) and as a function of the coverage $\big(w_i(A)\big)$ of the target shape (A). A distance is defined on the search space (*sharing* method) to get the individuals to be as far as possible one of each other (linked to the euclidian distance between the w_i fixed points). The w_i population then evolves so that the best individuals of the population get the best and most economical covering of the target image. $A = \cup w_i(A)$.

- **Constrained fixed points:** The fixed points x_i of individuals w_i are randomly chosen among the contour points of the target shape A in the initial generation[4]. Mutation and crossover are adapted in order to insure that the x_i always stay on the contour of A.

  Mutation of and individual w_i is for its fixed point x_i a uniform random shot among the contour points in a neighbourhood of fixed size, and a random modification of the tree structure for its function tree.

  Crossover between individuals w_i and w_j does not modify their fixed points, and is a classical GP crossover (subtree exchange) between their tree structures.

- **Local fitness:** a combination of two terms according to:

  1. *The position of $w_i(A)$ with respect to A.*
     As the w_i fixed points are constrained to belong to A. We can compute the image $w_i(A)$, in order to test the set $w_i(A)$ with respect to A.
     Let $\#[A]$ be the number of pixels of A, the term $F_1(w_i)$ is:

$$F_1(w_i) = \frac{1}{1 + \#[w_i(A) \backslash A]}$$

[4] This constraint is related to a conjecture by Michel DEKKING that there always exist solutions to the inverse problem where fixed points are on the edges of the target shape. This result has been proven in the case of affine IFS in [6].

Table 2: Fitness parameters for the inverse problem, using GA-Lib

Local fitness tuning	$\alpha = 0.4$
Mutation probabilities	
constant $\rightarrow$ constant	0.15 according to a Gaussian law of variance SIGMA
variable $\rightarrow$ constant	0.05 randomly chosen in $[-1, 1]$
constant $\rightarrow$ variable	0.06
variable $\rightarrow$ variable	0.08
function $\rightarrow$ function (same arity)	0.08
fixed points:	0.4, linearly decreasing with generation uniform random choice among contour pixels in a neighborhood of radius 4 pixels
Crossover probability	
PCROSS	1. for trees and fixed points
Sharing	
σ (Goldberg [10])	2*SIGMA
Population replacement scheme	
replacement percentage	50% Overlaping populations

$F_1(w_i)$ is maximum (and equals 1) if $w_i(A) \subset A$.

2. *The coverage of A with $w_i(A)$.*
   A term F_2 has also to be defined, that corresponds to the maximisation of the size of $w_i(A) \bigcap A$.

$$F_2(w_i) = \frac{\#[w_i(A) \bigcap A]}{\#[A]}$$

$F_2(w_i)$ is maximum (and equals 1) if $A \subset w_i(A)$.

The local fitness of the individual w_i is a linear combination of the previous terms.

$$F_{loc}(w_i) = (1 - \alpha)F_1(w_i) + \alpha F_2(w_i) \qquad (5)$$

This fitness represents an interpretation of the "collage" property of an IFS, i.e.: one searches for the set of best w_i's such that $A = \bigcup w_i(A)$. One also understands the benefit of a sharing scheme in order to have an economic coverage of A with the sets $w_i(A)$.

- **Contractivity constraints** are considered as in section 4.

- **Global fitness:**
  The N (to be determined with respect to the local fitness[5]) best individuals of the evolved population are evaluated via a toss-coin algorithm. The attractor $\mho$ of these best individuals is computed, the global fitness then is:

$$F_{glob} = \frac{1}{\#[\mho]} \Sigma_{x \in \mho} DIST(x) + \frac{\#[\mho \bigcap A]}{\#[A]}$$

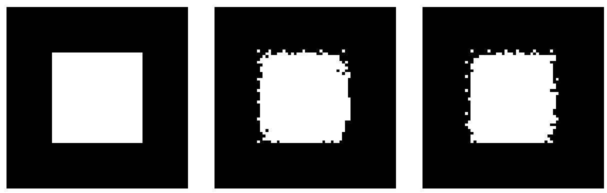
$DIST(x)$ is the pixel value of x in the distance image of target shape A[6].

F_{glob} is a measurement of the distance between $\mho$ and A. The first term of this sum represents the mean distance of the set $\mho$ to A (1 if $\mho \subset A$), the second term is 1 if $A \subset \mho$.

This global fitness is distributed on N best w_i, proportionally to their contribution to the target approximation in an additive manner.

$$F(w_i) = F_{loc}(w_i) + k_i F_{glob}$$

F_{glob} can also be used to stop the algorithm, i.e.: when the target is approximated with a fixed threshold.



Target	approx. 85.54% 22 generations 19 functions	approx. 85.06% 19 generations 20 functions

Figure 5: Two runs of the algorithm: 64x64 target, with a population size of 300 individuals, the algorithm is stopped as soon as $\frac{\#[\mho]}{\#[A]} > 85\%$.

Results obtained with the parameter setting of table 2, are presented in figures 5 and 6.

6 Conclusion

The aim of this paper was to show the interest of using optimisation strategies for evolutionary algorithms

[5] We select all the contractive individuals having a F_1 near 1, i.e.: whose fixed points are close to the target shape A. This set is then filtered by a simple clustering scheme in order to select only the best individuals of each cluster.

[6] A distance image is the transformation of a black & white image (the target shape A) into a grey-level one, where the level affected to each image point is a function of its distance to the original shape A. It can be easily computed by a simple algorithm (see [4]).

Target	approx. 60.46%	approx. 60.13%
	256x256 target	128x128 target
	1000 indiv.	2000 indiv.
	38 generations	64 generations
	15 functions	27 functions

Figure 6: Two runs of the algorithm: Dolphin target, the algorithm is stopped as soon as $\frac{\#[\tilde{U}]}{\#[A]} > 60\%$.

other than the usual direct implementation that identifies the fitness function to the function to be optimised. Of course an individual approach can only be used on specific problems such as the ones we presented here. The careful design of fitness functions and balance between local terms and global terms is crucial for the quality and efficiency of the method.

However, the examples we have exhibited in this paper show the benefit of individual strategies: for the inverse problem a rough approximation of the shape is obtained very rapidly while fine tuning are longer to obtain. In comparison to the "direct" implementation, one needs a reduced number of generations (and consequently a reduced number of fitness evaluations) to converge to an acceptable result[7]. An interesting experiment in the case of the inverse problem (that may also prove that "individual" approaches have still to be considered as "regular" Evolutionary algorithms) is to run the GP algorithm without the global fitness term : results are almost similar, the influence of the global fitness is small. The implementation of the individual approach we describe here differ thus from Credit Assigment Problems [10], where no information is available to measure the efficiency of individuals, except the one that comes from a global evaluation and that has to be dispatched on individuals.

We also show that Polar IFS is an interesting model that simplifies the manipulation of non-linear IFS. Future work on this topic concern:

- implementations of section 4 technique in an interactive manner for artistic generation of fractal images,
- exploitation of inverse problem for Polar IFS in the framework of physical structures optimisation.

[7]A precise comparison between these approaches is not straightforward : due to the difference of individuals and fitness functions structures in each approach, a comparison with respect to the number of generations or fitness evaluations is not convenient. A more precise analysis as well as an hybrid implementation (where individual and global GA collaborate) is a part of future works we intend to do on this topic

References

[1] M. Barnsley, S. Demko, J. Elton, and Geronimo J. Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities. *Georgia Tech. preprint.*

[2] M. Barnsley, V. Ervin, D. Hardin, and J. Lancaster, "Solution of an Inverse Problem for Fractals and Other Sets," *Proc. Natl. Acad. Sci. USA*, 83, 1986.

[3] M. F. Barnsley. Fractals Everywhere. *Academic Press,N Y*, 1988.

[4] G. Borgefors, "Distance Transformation in Arbitrary Dimension," *Computer Vision, Graphics, and Image Processing*, 27, 1984.

[5] Pierre Collet, Evelyne Lutton, Frédéric Raynal, and Marc Schoenauer, "Polar IFS and 'individual' Genetic Programming," *Technical report, INRIA Research Report*, 1999.

[6] F.M. Dekking, "Fractal Image Coding : Some Mathematical Remarkd on Its Limits and Its Prospects," chapter in *Fractal Image Encoding and Analysis*, Yuval Fisher, Ed, pages 117–133. *NATO ASI Series, Springer Verlag Series F : Computer and System Sciences*, Vol. 159, 1998.

[7] J. H. Elton. An ergodic theorem for iterated maps. *Georgia Tech. preprint*, 1986.

[8] Y. Fisher. Fractal image compression. *Siggraph 92 course notes*, 1992.

[9] B. Goertzel. Fractal image compression with the genetic algorithm. *Complexity International*, (1), 1994.

[10] David A. Goldberg, "Genetic Algorithms in Search, Optimization, and Machine Learning," *Addison-Wesley Publishing Company, inc., Reading, MA*, January 1989.

[11] D. P. Hardin. Hyperbolic Iterated Function Systems and Applications. *PhD thesis, Georgia Institute of Technology*, 1985.

[12] J. Hutchinson. Fractals and self-similarity. *Indiana University Journal of Mathematics*, 30:713–747, 1981.

[13] A.E. Jacquin. Fractal image coding: a review. *Proceedings of the IEEE*, 81(10), october 1993.

[14] Jean Louchet, "Obstacle Detection and Segmentation-Free Stereovision: an Evolutionary Approach," *Submitted to GECCO'99.*

[15] E. Lutton, J. Lévy Véhel, G. Cretin, P. Glevarec, and C. Roll, "Mixed IFS: Resolution of the Inverse Problem using Genetic Programming," *Complex Systems*, 9:375–398, 1995. (see also Inria Research Report No 2631.)

[16] G. Mantica and A. Sloan. Chaotic optimization and the construction of fractals : solution of an inverse problem, *Complex Systems* . (3):37–62, 1989.

[17] D. J. Nettleton and R. Garigliano. Evolutionary algorithms and a fractal inverse problem . *Biosystems*, (33):221–231, 1994. Technical note.

[18] Frédéric Raynal, Pierre Collet, Evelyne Lutton, and Marc Schoenauer, "Manipulation of IFS Attractors using Genetic Programming," *Submitted to CEC99.*

[19] Karl Sims, "Artificial evolution for computer graphics," *Computer Graphics*, 25(4):319–328, July 1991.

[20] L. Vences and I. Rudomin. Fractal compression of single images and image sequences using genetic algorithms. *The Eurographics Association*, 1994.

[21] E. R. Vrscay. Moment and collage methods for the inverse problem of fractal construction with iterated function systems . In *Fractal 90 Conference*, 1990. Lisbonne, June 6-8.

[22] E. R. Vrscay. Fractal Geometry and Analysis, *Chapter Iterated function Systems: theory, applications and the inverse problem*, pages 405–468. 1991.