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# Studies on the Dynamics of Ant Colony Optimization Algorithms

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**Daniel Merkle**  
Institute AIFB  
University of Karlsruhe  
D-76128 Karlsruhe, Germany  
merkle@aifb.uni-karlsruhe.de

**Martin Middendorf**  
Computer Science Group  
Catholic University of Eichstätt-Ingolstadt  
D-85072 Eichstätt, Germany  
martin.middendorf@ku-eichstaett.de

## Abstract

A deterministic model for Ant Colony Optimization (ACO) algorithms is proposed and used to study the dynamics of ACO. The model is based on the average expected behaviour of ants. The behaviour of ACO algorithms and the model are analysed for certain types of permutation problems. It is shown numerically that decisions of the ants are influenced in an intriguing way by the properties of the pheromone matrix. This explains why ACO algorithms show a complex dynamic behaviour. Simulations are done to compare the behaviour of the ACO model with the ACO algorithm. The results show that the model describes essential features of the dynamics of ACO algorithms.

## 1 INTRODUCTION

Ant Colony Optimization (ACO) has been applied successfully to several optimization problems (ACO was proposed in [1, 2]). Since ACO algorithms are based on sequences of random decisions of artificial ants which are usually not independent it is difficult to analyze the behaviour of ACO algorithms theoretically. Except from convergence proofs for types of ACO algorithms with a strong elite principle [3, 4, 12] not much theoretical work has been done. Usually ant algorithms have been tested on benchmark problems or real world problems. In this paper we propose and analyze a deterministic model for ACO algorithms and use it to derive exact results on optimization problems with a simple structure. The analytical results are complemented with empirical tests to compare computations done with the ACO model with test runs of the ACO algorithm.

Modelling has been done in the field of genetic algorithms (GAs) by several authors in order to better understand GAs behaviour. One line of modelling is to use an infinite population which is often easier to handle than a finite population since many properties of an infinite population do not fluctuate due to few random events [13, 14]. Another method is to characterize the population by few parameters (e.g., mean and variance of the fitness distribution of the population) which capture important aspects of the population instead of dealing with a concrete population (e.g., [9–11]). Mostly in these studies, GAs are modelled on problems which are simple but have some characteristic features of more complicated and real-world problems (e.g., “Royal Road” functions [8]).

The approach used in this paper is to define a deterministic model for ACO that is based on the expected decisions of the ants. In the model the pheromone update in every iteration is done by adding for each pheromone value the expected update of a random generation of ants.

In Section 2, we describe the permutation problems that are used in this paper. The ACO algorithm is described in Section 3 and the ACO model is defined in Section 4. In Section 5, we discuss how to apply the model to permutation problems. A fixed point analysis of pheromone matrices is done in Section 6. In Section 7, we analyze the dynamic behaviour of the ACO model. Simulation results are described in Section 8 and conclusions are given in Section 9.

## 2 PERMUTATION PROBLEMS

Although the general approach of our ACO model does not depend on a specific type of optimization problems we give a more elaborated description only for permutation problems. They are also used as test problems. In particular, we use the following type of permutation

problems. Given are  $n$  items  $1, 2, \dots, n$  and an  $n \times n$  cost matrix  $C = [c(ij)]$  with integer costs  $c(i, j) \geq 0$ . Let  $\mathcal{P}_n$  be the set of permutations of  $(1, 2, \dots, n)$ . For a permutation  $\pi \in \mathcal{P}_n$  let  $c(\pi) = \sum_{i=1}^n c(i, \pi(i))$  be the cost of the permutation. Let  $\mathcal{C} := \{c(\pi) \mid \pi \in \mathcal{P}_n\}$  be the set of possible values of the cost function. The problem is to find a permutation  $\pi \in \mathcal{P}_n$  of the  $n$  items that has minimal costs, i.e., a permutation with  $c(\pi) = \min\{c(\pi') \mid \pi' \in \mathcal{P}_n\}$ .

### 3 ACO ALGORITHM

An ACO algorithm consists of several iterations where in every iteration each of  $m$  ants constructs a solution for the optimization problem. It has to be mentioned that we can not consider here all the variants and improvements that have been proposed in recent years for ACO.

For the construction of a solution (here a permutation) every ant selects the items in the permutation one after the other. For the selection of an item the ant uses pheromone information which stems from former ants that have found good solutions. The pheromone information, denoted by  $\tau_{ij}$ , is an indicator of how good it seems to have item  $j$  at place  $i$  of the permutation. The matrix  $(\tau_{ij})_{i,j \in [1:m]}$  of pheromone values is called the pheromone matrix. In addition an ant may also use problem specific heuristic information. But since we want to study ACO algorithms in general and not for some specific problem we do not consider heuristics in this paper.

The next item is chosen by an ant from the set  $\mathcal{S}$  of items, that have not been placed so far, according to the following probability distribution (e.g. [2]) that depends on the pheromone values in row  $i$  of the pheromone matrix:  $p_{ij} = \tau_{ij} / \sum_{h \in \mathcal{S}} \tau_{ih}$ ,  $j \in \mathcal{S}$ .

Note that alternative methods where the ants do not consider only the local pheromone values have also been proposed [5, 7]. Before the pheromone update is done a certain percentage of the old pheromone evaporates according to the formula  $\tau_{ij} = (1-\rho) \cdot \tau_{ij}$ . Parameter  $\rho$  allows to determine how strongly old pheromone influences future decisions. Then, for every item  $j$  of the best permutation found so far some amount  $\Delta$  of pheromone is added to element  $\tau_{ij}$  of the pheromone matrix ( $i$  is the place of item  $j$ ). The algorithm stops when some stopping criterion is met, e.g. a certain number of generations has been done. For ease of description we assume that the sum of the pheromone values in every row and every column of the matrix is always one, i.e.,  $\sum_{i=1}^n \tau_{ij} = 1$  for  $j \in [1 : n]$  and  $\sum_{j=1}^n \tau_{ij} = 1$  for  $i \in [1 : n]$  and  $\Delta = \rho$ .

### 4 ACO MODEL

In the proposed ACO model the pheromone update of a generation of ants is done by adding to each pheromone value the expected update value. This means that the effect of an individual ant in a run is averaged out. Since the update values in the ACO algorithm are always only zero or  $\Delta = \rho$  the ACO model can only approximate the average behaviour of an ACO algorithm over more than one generation.

In order to determine the expected update for a random generation of ants the probabilities for the various decisions of the ants have to be determined. Let  $M = (\tau_{ij})$  be a pheromone matrix and let  $\sigma_{ij}$  be the probability that a random ant selects item  $j$  for place  $i$ . Clearly, this selection probability can be computed as described in the following. Let  $\mathcal{P}_n$  be the set of possible solutions, i.e. the set of permutations of  $(1, 2, \dots, n)$ . The probability to select a solution  $\pi \in \mathcal{P}_n$  is

$$\sigma_\pi = \prod_{i=1}^n \frac{\tau_{i,\pi(i)}}{\sum_{j=i}^n \tau_{i,\pi(j)}} \quad (1)$$

The probability that item  $i$  is put on place  $j$  is

$$\sigma_{ij} = \sum_{\pi \in \mathcal{P}_n} \sigma_\pi \cdot g(\pi, i, j)$$

where  $g(\pi, i, j) = 1$  if  $\pi(i) = j$  (otherwise  $g(\pi, i, j) = 0$ ).

Given a permutation problem  $P$  with corresponding cost matrix and pheromone matrix let  $\sigma_{ij}^{(m)}$  be the probability that the best of  $m$  ants in a generation selects item  $j$  for place  $i$ . Let  $\mathcal{P}_{min}(P, \pi_1, \dots, \pi_m)$  be the subset of permutations of  $\{\pi_1, \dots, \pi_m\}$  with minimal costs, i.e.,  $\mathcal{P}_{min}(P, \pi_1, \dots, \pi_m) = \{\pi_i, i \in [1 : m] \mid c(\pi_i) = \min\{c(\pi_j) \mid j \in [1 : m]\}\}$ . Probability  $\sigma_{ij}^{(m)}$  can be computed by

$$\sigma_{ij}^{(m)} = \sum_{(\pi_1, \dots, \pi_m), \pi_i \in \mathcal{P}_n} \left( \prod_{k=1}^m \sigma_{\pi_k} \right) \cdot g(\pi_1, \dots, \pi_m, i, j) \quad (2)$$

where  $g(\pi_1, \dots, \pi_m, i, j)$  equals

$$\frac{|\{\pi \in \mathcal{P}_{min}(P, \pi_1, \dots, \pi_m) \mid \pi(i) = j\}|}{|\mathcal{P}_{min}(P, \pi_1, \dots, \pi_m)|} \quad (3)$$

At the end of a generation the pheromone update is done in the ACO model by  $\tau_{ij} = (1 - \rho) \cdot \tau_{ij} + \rho \cdot \sigma_{ij}^{(m)}$ .

In the following an alternative way to compute the selection probabilities for the best of  $m$  ants is described. Let  $\mathcal{C}$  be the set of possible cost values for a permutation or in other words the set of possible solution qualities. Let  $\xi^{(m,x)}$  be the probability that the best of  $m$  ants in a generation finds a solution with quality  $x \in \mathcal{C}$ . Let  $\omega_{ij}^{(x)}$  be the probability that an ant which found a solution with quality  $x \in \mathcal{C}$  has selected item  $i$  for place  $j$ . Then

$$\sigma_{ij}^{(m)} = \sum_{x \in \mathcal{C}} \xi^{(m,x)} \cdot \omega_{ij}^{(x)} \quad (4)$$

An interesting aspect of this formula is that the pheromone update that is performed at the end of an iteration is obtained as a weighted sum over the possible solution qualities. For each (possible) solution quality the update value is determined by the probabilities for the decisions of a single ant when it chooses between all possible solutions with that same quality. The effect of the number  $m$  of ants is only that the weight of the different qualities in this sum changes. The more ants per iteration, the higher becomes the weight of the optimal quality.

We now consider a variant of the above situation that is needed in the next section. We introduce the concept of malus values. It is assumed that some ants in an iteration receive a malus value. This value is added to the cost of the permutation they found. An ant with a malus is allowed to update only when the cost of its solution plus the malus is better than the solution of every other ant plus its malus (in case it has a malus). Formally, for  $i \in [1 : m]$  let  $d_i \geq 0$  be the malus of ant  $i$ . We always assume that ant 1 has no malus, i.e.  $d_1 = 0$ . Let  $\xi^{m;x;d_2,\dots,d_m}$  be the probability that the best of  $m$  ants where ant  $i \in [1 : m]$  has a malus  $d_i$  has found a solution of quality  $x \in \mathcal{C}$ . Then

$$\sigma_{ij}^{(m;d_2,\dots,d_m)} = \sum_{x \in \mathcal{C}} \xi^{(m;x;d_2,\dots,d_m)} \cdot \omega_{ij}^{(x)} \quad (5)$$

## 5 ACO MODEL FOR RESTRICTED PERMUTATION PROBLEMS

Many real world problems consist of subproblems that are more or less independent from each other. In order to study the behaviour of ACO algorithms on such problems we model this in an idealized way. We consider restricted permutation problems which consist of several small independent problems. Define for a permutation problem  $P$  of size  $n_0$  a restricted permutation

problem  $P^q$  that consists of  $q$  independent instances of  $P$ . Formally, let  $C_1, C_2, \dots, C_q$  be the cost matrices of  $q$  instances of  $P$  and denote by  $c_{ij}^{(l)}$  element  $(i, j)$  of matrix  $C_l$ ,  $l \in [1 : q]$ . Then for problem  $P^q$  the item  $(l-1) \cdot n_0 + j$ ,  $l \in [1 : q]$ ,  $j \in [1 : n_0]$  can be placed only at places  $(l-1) \cdot n_0 + 1, (l-1) \cdot n_0 + 2, \dots, (l-1) \cdot n_0 + n_0$ . The cost to place item  $(l-1) \cdot n_0 + j$  at place  $(l-1) \cdot n_0 + h$  is  $c_{hj}^l$ . Let  $C$  be the corresponding cost matrix of the instance of problem  $P^q$  where  $c_{ij} = \infty$  when  $j$  is not of the form  $(l-1) \cdot n_0 + h$ . Note, that our definition of restricted permutation problems does not allow an ant to make a decision with cost  $\infty$ . We call  $P$  the elementary subproblem of  $P^q$  and the  $q$  instances of  $P$  that form an instance of  $P^q$  the elementary subinstances of  $P^q$ . We consider here only the case that all cost matrices  $C_1, C_2, \dots, C_q$  are equal, i.e.  $C = C_1 = C_2 = \dots = C_q$  for some cost matrix  $C$ . Then  $P^q$  is called homogeneous restricted permutation problem and the cost matrix of  $P^q$  is denoted by  $C^{(q)}$ .

In the following we show how the behaviour of the ACO algorithm for a (possibly inhomogeneous) restricted permutation problem  $P^q$  can be approximated using the behaviour of the ACO model for the elementary subproblem  $P$ . Consider an arbitrary of the  $q$  elementary subinstances of  $P^q$  — say the  $r$ th subinstance — and the quality of the solutions that  $m$  ants in an iteration have found on the other elementary subinstances (which form an instance of problem  $P^{q-1}$ ). Without loss of generality assume that the quality of the solution found by ant  $i$  is at least as good as the solution found by ant  $i+1$ ,  $i \in [1 : m-1]$ . Let  $d_{max}$  be the maximum difference between two values in cost matrix  $C_r$  of the  $r$ th subproblem, i.e.  $d_{max} := \max\{c_{ij}^{(r)} \mid i, j \in [1 : n_0]\} - \min\{c_{ij}^{(r)} \mid i, j \in [1 : n_0]\}$ . Let  $d_i$ ,  $i \in [2 : m]$  be the minimum of  $d_{max} + 1$  and the difference of the cost of the permutation found by ant  $i$  on  $P^{q-1}$  minus the cost of the permutation found by ant 1 on  $P^{q-1}$ . Our assumption implies  $0 \leq d_2 \leq \dots \leq d_m$ . Define  $\phi^{m;d_2,\dots,d_m}$ ,  $0 \leq d_i \leq d_{max} + 1$ ,  $i \in [2 : m]$  as the probability that for  $m$  ants on problem  $P^{q-1}$  the difference of the costs of the solutions found by the  $i$ th best ant and the best ant is  $d_i$  when  $d_i \leq d_{max}$  and when  $d_i = d_{max} + 1$  it is the probability that this difference is  $> d_{max}$ ,  $i \in [2 : m]$ . Let  $\mathcal{D}$  be the set of all vectors  $(d_2, \dots, d_m)$  with integers  $d_2 \leq \dots \leq d_m$ ,  $0 \leq d_i \leq d_{max} + 1$ ,  $i \in [2 : m]$ . Then for the  $r$ th elementary subproblem of  $P^q$  we obtain  $\sigma_{ij}^{(m)}$  equals

$$\sum_{(d_2,\dots,d_m) \in \mathcal{D}} \phi^{(m;d_2,\dots,d_m)} \cdot \sigma_{ij}^{(m;d_2,\dots,d_m)} = \sum_{x \in \mathcal{C}} w_x \cdot \omega_{ij}^{(x)}$$

with  $w_x = \sum_{(d_2, \dots, d_m) \in \mathcal{D}} \phi^{(m; d_2, \dots, d_m)} \cdot \xi^{(m; x; d_2, \dots, d_m)}$ . This shows that the effect of the subproblem  $P^{q-1}$  on the remaining subinstance  $P_r$  of  $P^q$  is to change the weights between the influence of the different solution quality levels when compared to formula 4 for solving only the subproblem  $P = P_r$ .

We study the case of  $m = 2$  ants in more detail. For problem  $P^q$  let  $p_{d>y}$  and  $p_{d=y}$  be the probabilities that the absolute value of the difference of the solution quality of two ants on the smaller problem  $P^{q-1}$  is  $> y$  respectively  $= y$ . Then the probability  $\sigma_{(l-1)n_0+h, (l-1)n_0+j}^{(2)}$  to select item  $j$  of the  $l$ th elementary subproblem equals

$$p_{d>d_{max}} \cdot \sigma'_{hj} + \sum_{y=1}^{d_{max}} p_{d=y} \cdot \sigma'_{hj}^{(2;y)} + p_{d=0} \cdot \sigma'_{hj}^{(2)}$$

where  $\sigma'$  refers to probabilities for the elementary subinstance  $P$ . This equation shows the interesting fact that part of the probability to select the item is just the probability  $\sigma'_{hj}$  of a single ant to select item  $j$  at place  $h$  for the elementary subproblem  $P$ . This is the case when the quality of the solutions of both ants differ by more than  $d_{max}$ . When the qualities of both solutions are the same the probability  $\sigma'_{hj}^{(2)}$  to select item  $j$  at place  $h$  equals the probability that the better of two ants on problem  $P$  selects item  $j$  at place  $h$ . All other cases correspond to the situation that one of two ants on problem  $P$  has a malus.

The larger the number  $q$  of subproblems is the larger becomes the probability  $p_{d>d_{max}}$ . An important consequence is that the (positive) effect of competition between the two ants for finding good solutions becomes weaker and a possible bias in the decisions of a single ant has more influence.

Let  $q_{d=y}$  ( $q_{d=y}^{(q-1)}$ ) be the probability that the difference of the solution quality found by the first ant minus the solution quality found by the second ant on subproblem  $P$  (respectively  $P^{q-1}$ ) is  $y$  (here we do not assume that the first ant finds the better solution). The value of this difference on subproblem  $P^{q-1}$  can be described as the result of a generalized one-dimensional random walk of length  $q-1$ . Define  $I_y$  as the set of tuples  $(k_{-d_{max}}, k_{-d_{max}+1}, \dots, k_{d_{max}-1}, k_{d_{max}})$  with  $q-1 = \sum_{i=-d_{max}}^{d_{max}} k_i$ ,  $y = \sum_{i=-d_{max}}^{d_{max}} k_i \cdot d_i$  where  $k_i$  is the number of elementary subinstances of  $P^{q-1}$  where the difference between the first and the second ant is  $i \in [-d_{max} : d_{max}]$ . Then  $q_{d=y}^{(q-1)}$  can be computed as follows

$$\sum \frac{(q-1)!}{k_{-d_{max}}! \cdot \dots \cdot k_{d_{max}}!} \cdot q_{d=-d_{max}}^{k_{-d_{max}}} \cdot \dots \cdot q_{d=d_{max}}^{k_{d_{max}}}$$

where the sum is over  $(k_{-d_{max}}, \dots, k_{d_{max}}) \in I_y$ . Clearly,  $p_{d=0} = q_{d=0}^{(q-1)}$  and due to symmetry, for  $y \neq 0$   $p_{d=y} = 2 \cdot q_{d=y}^{(q-1)}$ . The remarks on analysing the ACO model for  $m = 2$  ants can be extended to  $m \geq 3$ .

As an example consider the following problem  $P_1$  with cost matrix

$$C_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \quad (6)$$

The possible solution qualities for problem  $P_1$  are 0, 2, and 4 and the optimal solution is to put item  $i$  on place  $i$  for  $i \in [1 : 3]$ . Hence  $\sigma_{3i+j, 3i+h}^{(2)} = p_{d>4} \cdot \sigma'_{hj} + \sum_{y=2,4} p_{d=y} \cdot \sigma'_{hj}^{(2;y)} + p_{d=0} \cdot \sigma'_{hj}^{(2)}$ . Consider the following pheromone matrix for  $P_1$

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \end{pmatrix} \quad (7)$$

Then the probability for an ant to put, e.g., item 2 on place 2 can be computed as  $\sigma_{22} = 0.1 \cdot 0.1 / (0.1 + 0.3) + 0.6 \cdot 0.1 / (0.1 + 0.6) \approx 0.111$ . The matrix of selection probabilities for one ant on problem  $P_1$  is

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \approx \begin{pmatrix} 0.1 & 0.3 & 0.6 \\ 0.714 & 0.111 & 0.175 \\ 0.186 & 0.589 & 0.225 \end{pmatrix}$$

Since the optimal solution is to place item  $i$  on place  $i$  for  $i \in [1 : 3]$  it seems likely that the corresponding selection probabilities are larger with two ants per iteration compared to the case of a single ant in an iteration. But our example shows that this is not necessary. The probability to place item 2 on place 2 is  $\sigma_{22}^{(2)} = 0.109$  and slightly smaller than  $\sigma_{22} = 0.111$ . When one of two ants has a malus the selection probabilities are mostly in between the case of two ants per iteration and a single ant per iteration. But again, the probability to place item 2 on place is a counterexample:  $\sigma_{22}^{(2)} < \sigma_{22} < \sigma_{22}^{(2;2)}$ . Although not true in every case, it can be observed that the selection probabilities for the better ant become more similar to the matrix of the selection probabilities for a single ant the higher the malus is.

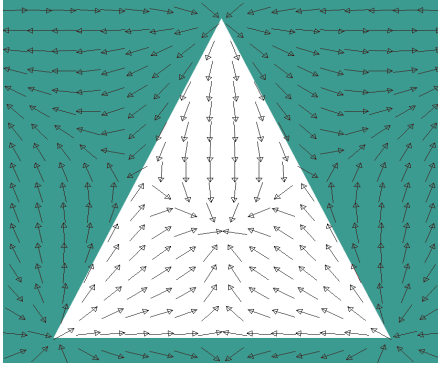


Figure 1: Direction of the vector field for changes of pheromone vectors  $(\tau_{21}, \tau_{22}, \tau_{23})$  for the ACO model with a single ant and  $\tau_{11} = \tau_{12} = \tau_{13} = 1/3$ ; possible pheromone vectors lay in the white area; distance from right (bottom, left) line:  $\tau_{21}$  ( $\tau_{22}, \tau_{23}$ )

## 6 FIXED POINTS

Since the pheromone values of an ACO algorithm reflect the frequencies of decisions that resulted in good solutions it is desirable that the selection probabilities used by an ant are equal to their corresponding pheromone values. As observed in [6] this might often not be the case since the decisions of an ant are not independent. We say that there is a selection bias when the probability of a random ant to choose an item is different from the corresponding pheromone value. A pheromone matrix where the probability of a random ant to choose an item is the same as the corresponding pheromone value, i.e.,  $\tau_{ij} = \sigma_{ij}$ , for all  $i, j$ , is called fixed point matrix. A fixed point matrix can change in the ACO model only for  $m \geq 2$  ants. The question arises which matrices are fixed point matrices for a permutation problem.

As an example consider a permutation problem of size  $n = 3$  where the pheromone matrix  $(\tau_{ij})_{i,j \in [1:3]}$  is defined by the values  $\tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}$ . Clearly, the selection probabilities for the items in the first row are always equal to their corresponding pheromone values. It remains to determine the probability of choosing the first and the second item in the second row. The selection probabilities for these items are

$$\sigma_{21} := \frac{\tau_{12} \tau_{21}}{1 - \tau_{22}} + \frac{(1 - \tau_{11} - \tau_{12}) \tau_{21}}{\tau_{21} + \tau_{22}}$$

$$\sigma_{22} := \frac{\tau_{11} \tau_{22}}{1 - \tau_{21}} + \frac{(1 - \tau_{11} - \tau_{12}) \tau_{22}}{\tau_{21} + \tau_{22}}$$

The solutions of  $\sigma_{21} - \tau_{21} = 0$  and  $\sigma_{22} - \tau_{22} = 0$  show

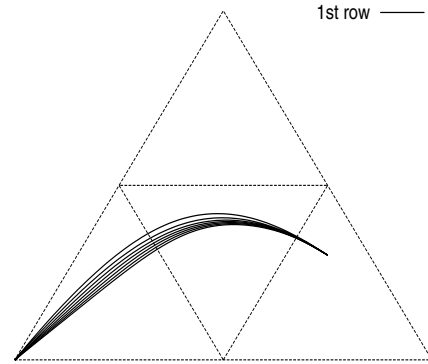


Figure 2: ACO model for  $P_1^q$ ,  $q = 2, 4, 8, 16, 32, 64, 128$ ,  $m = 2$ ; change of pheromone values  $\tau_{11}, \tau_{12}, \tau_{13}$  (first row of corresponding fixed point matrix is identical): starting point at  $(\tau_{11}, \tau_{12}, \tau_{13}) = (0.1, 0.3, 0.6)$

that the fixed points depend only on the pheromone values in the first row of the pheromone matrix:

1.  $\tau_{21} = 0, \tau_{22} = (\tau_{11} + \tau_{12} - 1)/(\tau_{11} - 1)$
2.  $\tau_{21} = (\tau_{11} + \tau_{12} - 1)/(\tau_{12} - 1), \tau_{22} = 0$
3.  $\tau_{21} = (\tau_{11})/(\tau_{11} + \tau_{12}), \tau_{22} = (\tau_{12})/(\tau_{11} + \tau_{12})$
4.  $\tau_{21} = 1 - 2 \cdot \tau_{11}, \tau_{22} = 1 - 2 \cdot \tau_{12}$

Analysing the eigenvalues of the Jacobian matrix of  $[\sigma_{21} - \tau_{21}, \sigma_{22} - \tau_{22}]$  the stability of the fixed points was determined. For every pair of possible values  $\tau_{11}$  and  $\tau_{12}$  exactly one of the fixed points is stable and attracting in the range of possible pheromone values. The cases (1), (2), and (3) are symmetric: for  $\tau_{11} > 0.5$  the fixed point (1) is stable, for  $\tau_{12} > 0.5$  the fixed point (2) is stable, and for  $1 - \tau_{11} - \tau_{12} > 0.5$  the fixed point (3) is stable. In every other case the fixed point (4) is stable. Thus, there exists always exactly one stable fixed point matrix. Some of the three unstable fixed points might lay outside of the allowed parameter range  $\tau_{21}, \tau_{22}, \tau_{23} \in (0, 1), \tau_{21} + \tau_{22} + \tau_{23} = 1$ .

The directions of the vector field for changes of pheromone vector  $(\tau_{21}, \tau_{22}, \tau_{23})$  when  $(\tau_{11}, \tau_{12}, \tau_{13}) = (1/3, 1/3, 1/3)$  are shown in Figure 1. In this case the vector field is symmetric with respect to rotations of 60 degree around the fixed point  $(\tau_{21}, \tau_{22}, \tau_{23}) = (1/3, 1/3, 1/3)$ . It is interesting to observe that in some areas of the vector field there are points  $(\tau_{21}, \tau_{22}, \tau_{23})$  with a value  $\tau_{2i} > 1/3$  for  $i \in [1 : 3]$  that becomes even larger. This shows that the effects of the selection bias can be complex even for small problems.

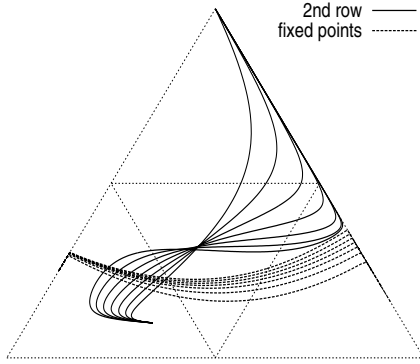


Figure 3: ACO model for  $P_1^q$ ,  $q = 2, 4, 8, 16, 32, 64, 128$ ,  $m = 2$ ; change of pheromone values  $\tau_{21}, \tau_{22}, \tau_{23}$  and fix point matrix: starting at  $(\tau_{21}, \tau_{22}, \tau_{23}) = (0.6, 0.1, 0.3)$

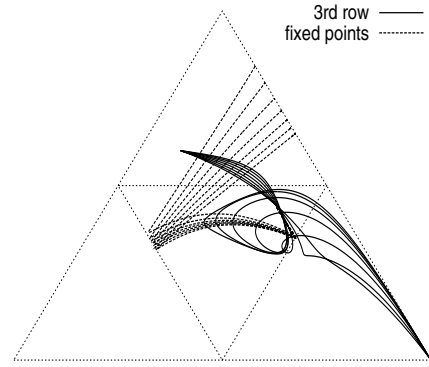


Figure 4: ACO model for  $P_1^q$ ,  $q = 2, 4, 8, 16, 32, 64, 128$ ,  $m = 2$ ; change of values  $\tau_{31}, \tau_{32}, \tau_{33}$  and fix point matrix: starting at  $(\tau_{31}, \tau_{32}, \tau_{33}) = (0.3, 0.6, 0.1)$

## 7 DYNAMIC BEHAVIOUR

To study the dynamic behaviour of the ACO model consider problem  $P_1$  with cost matrix given in (6) in Section 4. Figures 2-4 show the dynamic behaviour of the ACO model for  $P_1^q$  with different values of  $q$  and  $m = 2$  ants. The pheromone evaporation  $\rho$  is 0.1 and the initial pheromone matrix is the same as in (7).

In contrast to the situation of one ant where the model converges to its stable selection fixed point the model converges for this example to the optimal solution. During convergence the actual position of the selection fixed point has a strong influence on the system. Note that for the first row of the matrix the pheromone values always equal the corresponding selection probabilities. Hence all dynamic in the first row is only due to competition (and not due to selection bias). Therefore, the pheromone values in Figure 2 approach the optimal values on an almost straight path. This is different for the pheromone vectors of row 2 and 3 (see figures 3, 4) where the stable selection fixed point has a large influence and the system moves often more in direction of the stable selection fixed point than in direction to the optimal solution. In Figure 4 paths with  $q \geq 8$  contain a loop that is clearly influenced by the turn of the selection fixed point. The larger  $q$  the stronger is the deviation from a straight line because a high number of elementary subproblems leads to a small influence of competition (see Section 5).

In order to investigate the relative influence of selection, pure competition, and weak competition (where one ant has a malus) we computed the probabilities for the possible differences in solution quality between the two ants on the smaller problem  $P_1^{q-1}$ . Recall that

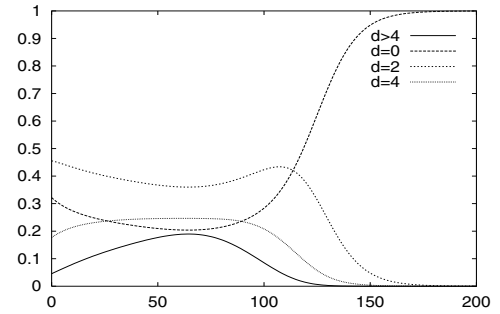


Figure 5: ACO model for  $P_1^q$ ,  $m = 2$ ,  $q = 4$ ; change of  $p_{d=y}$  on  $P_1^{q-1}$ :  $p_{d=0}, p_{d=2}, p_{d=4}, p_{d>4}$

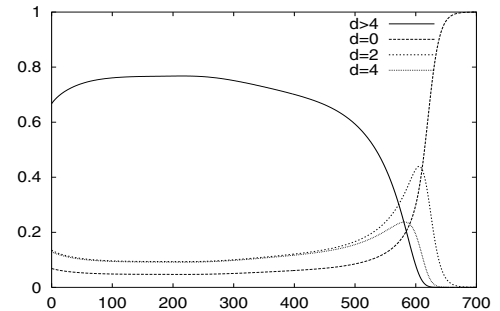


Figure 6: ACO model for  $P_1^q$ ,  $m = 2$ ,  $q = 64$ ; change of  $p_{d=y}$  on  $P_1^{q-1}$ :  $p_{d=0}, p_{d=2}, p_{d=4}, p_{d>4}$

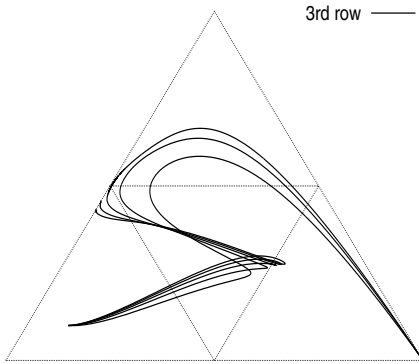


Figure 7: ACO model for problem  $P_2^q$ ,  $m = 2$ ,  $q = 6, 10, 14, 18, 22$ , initial matrix defined by  $(\tau_{11} = 0.1, \tau_{12} = 0.2, \tau_{21} = 0.1, \tau_{22} = 0.7)$ ; change of pheromone values  $\tau_{31}, \tau_{32}, \tau_{33}$

the solution quality for the elementary subproblem  $P_1$  can be 0, 2, or 4. Figures 5, 6 show the probabilities  $p_{d=0}, p_{d=2}, p_{d=4}, p_{d>4}$  for a small and a large number of subproblems. For the large number the cases  $p_{d>4}$ , respectively  $p_{d_1>4, d_2>4}$ , which correspond to selection by a single ant on the elementary subproblem have a probability of more than 50% over most parts of the run. Only when the ACO model starts to converge the model is driven more by competition (as suggested by the analysis of the dynamics of the pheromone values).

To show that the selection bias can be so strong that the ACO model is not able to find the optimal solution consider the following small problem  $P_2$  with cost matrix  $(c_{ij})$  where  $c_{ii} = 0, c_{13} = 100$  and all the other values are  $c_{ij} = 1, i, j \in [1 : 3]$ .

Figure 7 shows the behaviour of the ACO model for  $P_2^q$  with initial pheromone values  $\tau_{11} = 0.1, \tau_{12} = 0.2, \tau_{21} = 0.1, \tau_{22} = 0.7$  for different values of  $q$ . For  $q = 6, 10, 14$  subproblems the system converges to the optimal solution. But for larger numbers  $q = 18, 22$  the influence of the selection bias is so high that the system converges to a non-optimal solution with  $(\tau_{11}, \tau_{12}, \tau_{13}) = (x, 1 - x, 1), (\tau_{21}, \tau_{22}, \tau_{23}) = (0, 0, 1)$  and  $(\tau_{31}, \tau_{32}, \tau_{33}) = (1 - x, x, 0)$ . Even for small numbers the system is driven by a selection bias but competition becomes stronger early enough to change the direction of convergence to an non-optimal solution. We tested the system also for all 666 matrices with a feasible combination of pheromone values  $\tau_{ij} \in 0.1, 0.2, \dots, 0.9$ , for  $i, j \in [1 : 3]$ . Even for the small problem  $P_2^2$  the optimal solution can not be found for 83 of the 666 different initial matrices. This number increases up to 296 for  $P_2^{60}$ .

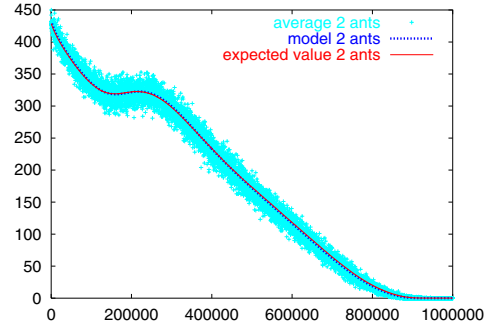


Figure 8: Solution quality on problem  $P_1^{128}$  with  $m = 2$  over 1000000 (respectively 1000) iterations: average  $m$  ants: average quality found by  $m$  ants in the iteration of ACO algorithm; expected value  $m$  ants: expected quality found by a random ant on the pheromone matrix in the generation of ACO algorithm; model  $m$  ants: average quality for ACO model

## 8 SIMULATION RESULTS

Since we have to consider single runs of the ACO algorithm in our simulation a very small value of  $\rho = 0.0001$  was chosen for the algorithm. For the ACO model  $\rho = 0.1$  was used. We compare then iteration  $t$  of the model with iteration  $1000t$  of the algorithm. Note that this establishes an additional difference between the model and the algorithm.

Figure 8 shows the behavior of the algorithm and model on problem  $P_1^q$  with  $q = 128$ . The solution quality found by a random ant of the ACO model is nearly the same as the expected behaviour of an ant in the ACO algorithm (in the figure the corresponding curves are nearly identical). The observed average solution quality of the ACO algorithm found by  $m = 2$  ants fluctuates around the solution quality that can be expected from the pheromone matrix in that generation. It is interesting that the expected solution quality of the ACO model and algorithm can become worse during the run (This effect is not the result of disadvantageous random decisions but is predicted by the model).

Figure 9 shows the results of the ACO algorithm on problems  $P_1^q$  with different values for  $q$ . Since every curve stems from one run of the algorithm only (it is not clear how to average in a reasonable way) the curves are not very smooth. Nevertheless the figure shows when compared to figures 3,4 that the ACO model predicts very well the development of the pheromone values of the ACO algorithm.

Of course not all aspects of ACO algorithms behaviour

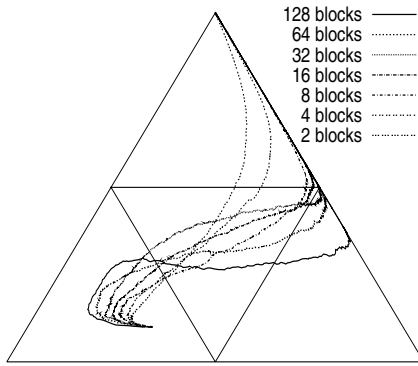


Figure 9: ACO algorithm for  $P_1^q$ ,  $q = 2, 4, 8, 16, 32, 64, 128$ ,  $m = 2$ : change of pheromone values  $\tau_{21}$ ,  $\tau_{22}$ ,  $\tau_{23}$  starting at  $(\tau_{21}, \tau_{22}, \tau_{23}) = (0.6, 0.1, 0.3)$

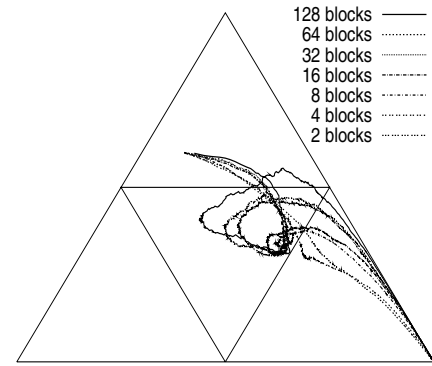


Figure 10: ACO algorithm for  $P_1^q$ ,  $q = 2, 4, 8, 16, 32, 64, 128$ ,  $m = 2$ : change of pheromone values  $\tau_{31}$ ,  $\tau_{32}$ ,  $\tau_{33}$  starting at  $(\tau_{31}, \tau_{32}, \tau_{33}) = (0.3, 0.6, 0.1)$

can be observed in the ACO model. As an example consider the restricted homogeneous permutation problem  $P^q$  where the ACO model shows the same behaviour on each of the elementary subproblems. The ACO algorithm in contrast behaves differently on the elementary subproblems due to random effects.

## 9 CONCLUSION

A deterministic model for ACO algorithms was proposed that uses a pheromone update mechanism based on the expected decisions of the ants. An interesting feature of the model is that it describes the behaviour of ACO algorithms through a combination of situations with different strength of competition between the ants. A fixed point analysis of the models pheromone matrices has given insights into the occurrence of biased decisions by the ants. It was shown analytically that the fixed points in the state space of the system have a strong influence on its optimization behaviour. Simulations have shown that the model accurately describes essential features of the dynamic behaviour of ACO algorithms.

## References

- [1] Dorigo, M. (1992). Optimization, Learning and Natural Algorithms (*in Italian*). PhD thesis, Dipartimento di Elettronica, Politecnico di Milano, Italy.
- [2] Dorigo, M., Maniezzo, V., and Colorni, A. (1996). The Ant System: Optimization by a Colony of Cooperating Agents. *IEEE Trans. Systems, Man, and Cybernetics - Part B*, 26:29–41.
- [3] Gutjahr, W. (2000). A graph-based Ant System and its convergence, *Future Generation Computer Systems*, 16:873–888.

- [4] Gutjahr, W. (2002). ACO algorithms with guaranteed convergence to the optimal solution. *Information Processing Letters*, 82:145–153.
- [5] Merkle, D., and Middendorf, M. (2000). An Ant Algorithm with a new Pheromone Evaluation Rule for Total Tardiness Problems. In Cagnoni, S., et al. (Eds.) *Real-World Applications of Evolutionary Computing*, LNCS 1803, 287–296, Springer.
- [6] Merkle, D., and Middendorf, M. (2001). A New Approach to Solve Permutation Scheduling Problems with Ant Colony Optimization. In Boers, E. J. W., et al. (Eds.) *Applications of Evolutionary Computing*, LNCS 2037, 213–222, Springer.
- [7] Merkle, D., and Middendorf, M. (2001). On the Behaviour of Ant Algorithms: Studies on Simple Problems. In *Proceedings of the 4th Metaheuristics International Conference (MIC'2001)*, Porto, 573–577.
- [8] Mitchell, M., and Forrest, S. (1997). Royal Road Functions. In Bäck, T., Fogel, D.B. and Michalewicz, Z. (Eds.), *Handbook of Evolutionary Computation*. B2.7:20–25, Oxford University Press, Oxford.
- [9] van Nimwegen, E., Crutchfield, J.P., Mitchell, M. (1999). Statistical dynamics of the royal road genetic algorithm. *Theor. Comp. Sci.*, 229:41–102.
- [10] Prügel-Bennett, A., and Rogers, A. (2001). Modelling GA Dynamics. In Kallel et al. (Eds.), *Theoretical Aspects of Evolutionary Computing*, Springer, 59–86.
- [11] Prügel-Bennett, A, and J.L. Shapiro (1997). An Analysis of Genetic Algorithms Using Statistical Mechanics. *Physica D*, 104:75–114.
- [12] Stützle, T. and Dorigo, M. (2000). A short convergence proof for a class of ACO algorithms. TR 2000-35, IRIDIA, Université Libre de Bruxelles, Belgium.
- [13] Vose, M.D., *The Simple Genetic Algorithm: Foundations and Theory* (1999). MIT Press, Cambridge, MA.
- [14] Vose, M.D., Liepins, G.E. (1991). Punctuated equilibria in genetic search. *Complex Systems*, 5:31–44.