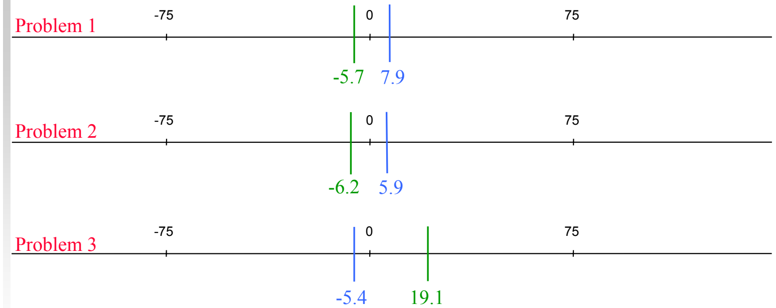
An Introduction to Statistical Analysis for Evolutionary Computation

Compiled and Written by
Mark Wineberg and Steffen Christensen

1

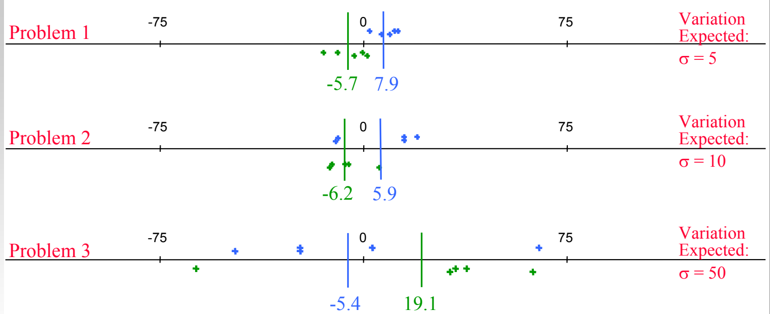
Sampling From Two Normal Distributions



System 1 Avg. | System 2 Avg.

Number of Runs = 5

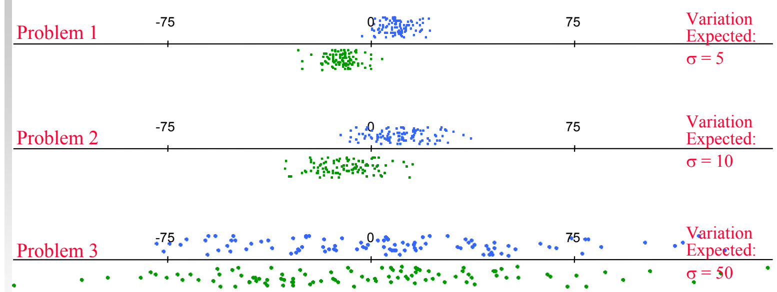
Sampling From Two Normal Distributions



True Avg. | -10 | +10

Number of Runs = 5

Sampling From Two Normal Distributions

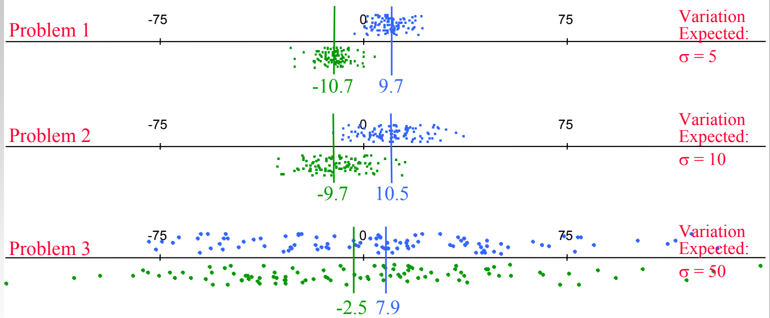


True Avg. | -10 | +10

Number of Runs = 100

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Sampling From Two Normal Distributions



True Avg.	-10	+10

Number of Runs = 100

Review of Simple Statistical Concepts Part 1

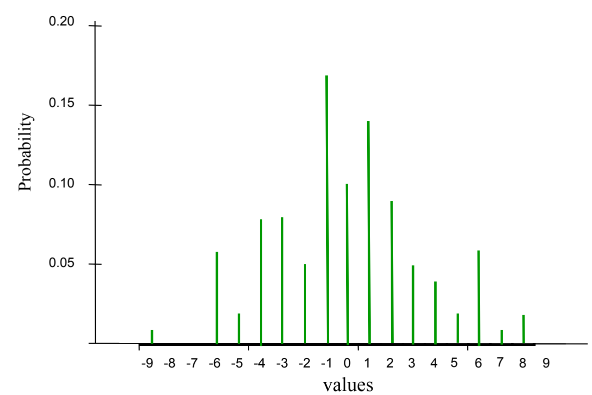
Random Variables
Common Probability Distributions

6

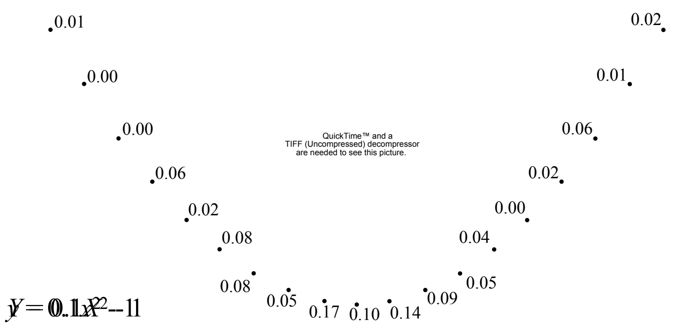
Random Variables

- Regular variables
  - Represents one or more values
  - Used in equations or inequalities that places restrictions on what values the variable can hold
- Random variables
  - Same as regular variables with one addition
    - Each value is associated with a probability of occurring

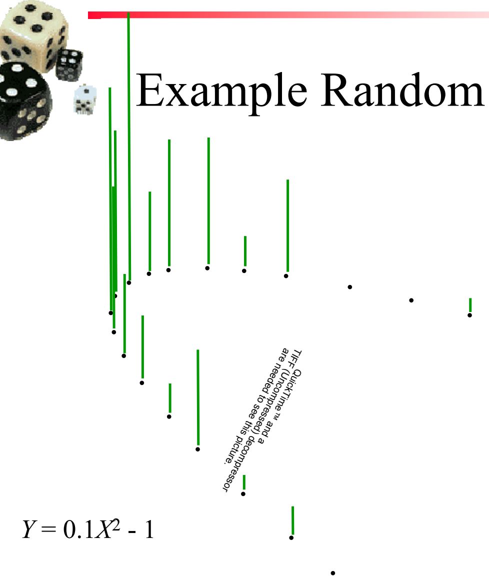
Probability Function



Example Variables



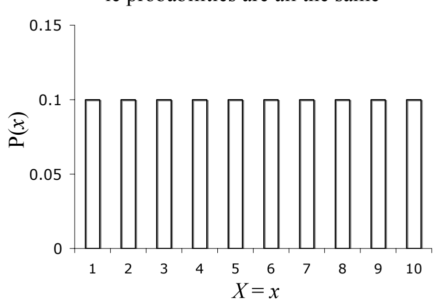
Example Random Variables



$Y = 0.1X^2$ - 1

Common Probability Distributions

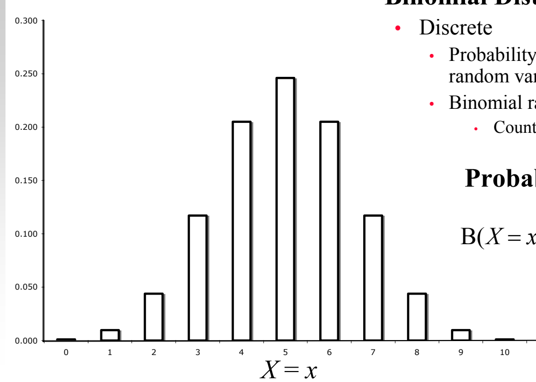
- **Uniform Distribution**
  - Discrete
    - All values are equally like
    - ie probabilities are all the same



X	P(X)
1	0.1
2	0.1
3	0.1
4	0.1
5	0.1
6	0.1
7	0.1
8	0.1
9	0.1
10	0.1

Common Probability Distributions

B(*X*=*x*; *n*=10, p=0.5)



- **Binomial Distribution**
  - Discrete
    - Probability associated with a binomial random variable
    - Binomial random variable
      - Counts binomial (two state) events

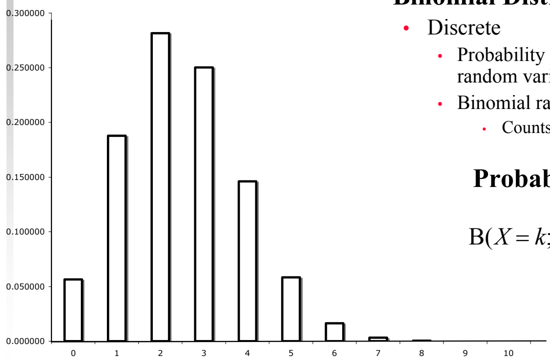
Probability Function

$$B(X = x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Common Probability Distributions

B(X=x; n=10, p=0.25)



$X = x$

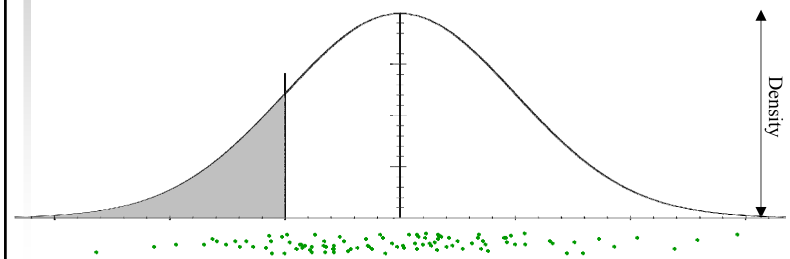
- **Binomial Distribution**
  - Discrete
    - Probability associated with a binomial random variable
    - Binomial random variable
      - Counts binomial (two state) events

Probability Function

$$\mathrm{B}(X = k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

Probability Density Function

- The greater the density of observed outcomes, the larger the density function's value
- Probability = integration of the density function within an interval
  - Note: probability for a point value is 0



Common Probability Distributions

- **Normal (Gaussian) Distribution**
  - Continuous
    - AKA Bell Curve
    - Most common continuous distribution found in nature (we will see why later)

$$P(X = x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$X \sim N(\mu, \sigma^2)$$

Standard Normal Distribution

$$X \sim N(0, 1)$$

QuickTime™ and a TIFF (Uncompressed) decompressor are needed to see this picture.

Common Probability Distributions

- **Linearity**
  - Any linear combination of normally distributed random variable is normally distributed

If $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ then

$$X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

$$X - Y \sim N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

In General

$$Y = \sum_{i=1}^{n} a_i X_i$$

Normally distributed

QuickTime™ and a TIFF (Uncompressed) decompressor are needed to see this picture.

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Review of Simple Statistical Concepts Part 2

Means and Variances

17

Mean vs Average

Expectation: taking the sum of the values of a random variable weighted by the probability of their occurrence
- result called *expected value* or *mean*

$$\mu = \mathrm{E}(X) = \sum_{i=1}^{n} p_i \cdot x_i \quad \langle x_i, p_i \rangle \in X \quad \sum_{i=1}^{n} p_i = 1$$

Average: the straight sum of the values of a population (a set that allows duplicates) divided by the number of values *n* in the population

$$\overline{v} = \mathrm{Avg}(P) = \frac{1}{n}\sum_{i=1}^{n} v_i \quad v_i \in P$$

If we have a uniform probability distribution mean = average

Properties of Expectations

Linearity

$$\mathrm{E}(b) = b$$

$$\mathrm{E}(aX + b) = a\,\mathrm{E}(X) + b$$

$$\mathrm{E}(aX + bY) = a\,\mathrm{E}(X) + b\,\mathrm{E}(Y)$$

where *a, b, c* are real numbers

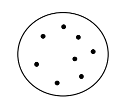
Composition

$$\mathrm{E}(\mathrm{E}(X)) = \mathrm{E}(\mu) = \mu = \mathrm{E}(X)$$

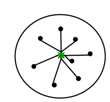
Similarly assuming that the expected value of f(X) is defined and is equal to the value denoted ϕ

$$\mathrm{E}(\mathrm{E}(f(X))) = \mathrm{E}(\phi) = \phi = \mathrm{E}(f(X))$$

Variation in a Population



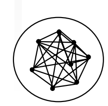
- Basic question:
  - How much variation is there in the population?
- Various Possibilities:
  - 1) How different are the values from the average value:
    - Can use L_1-norm



$$\mathrm{Var}_1(P) = \frac{1}{n}\sum_{i=1}^{n} |v_i - \overline{v}|$$

    - Can use L_2-norm
      - has nice mathematical properties when dealing with real values
      - called *variance*

$$\mathrm{Var}_2(P) = \frac{1}{n}\sum_{i=1}^{n} (v_i - \overline{v})^2$$



  - 2) Pair-wise Diversity

$$\mathrm{Div}_2(P) = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} (v_i - v_j)^2$$

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Variance of a Random Variable

$\text{var}(X) = E((X-\mu)^2)$ A.K.A. *the mean squared deviation*

var(X) can be written as σ_X^2 or σ^2

Standard Deviation

- Variance is measured in unit^2
- So the standard deviation is

$$\sigma = \sqrt{\text{var}(X)} = \sqrt{E((X-\mu)^2)}$$

- Statistical values are usually **reported** in terms of σ
- Most statistics are **computed** use variance

Various Variances

- Variance $\text{var}(X) = E((X-\mu)^2)$
- Variance of a Population $\text{var}(X) = \frac{1}{n}\sum_{i=1}^{n}(v_i - \bar{v})^2$
- Sample Variance $\text{var}(X) = \frac{1}{n-1}\sum_{i=1}^{n}(v_i - \bar{v})^2$

Variance Properties

Basic Properties

1) Variance is never negative
   Because the squares are positive or zero
2) If all elements of X are equal then var(X) = 0
   For example, the variance of 2, 2, 2, 2 is 0
3) If some elements of X are unequal then the var(X) > 0

Linear Transformations

$$\text{var}(aX+b) = a^2\,\text{var}(X)$$

Note: It follows that the variance in independent of the mean since

$$\text{var}(X-\mu) = \text{var}(X)$$

Basic Statistical Tests

Point Estimation: Finding the Mean Using Confidence Intervals

24

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

What Are We Interested In?

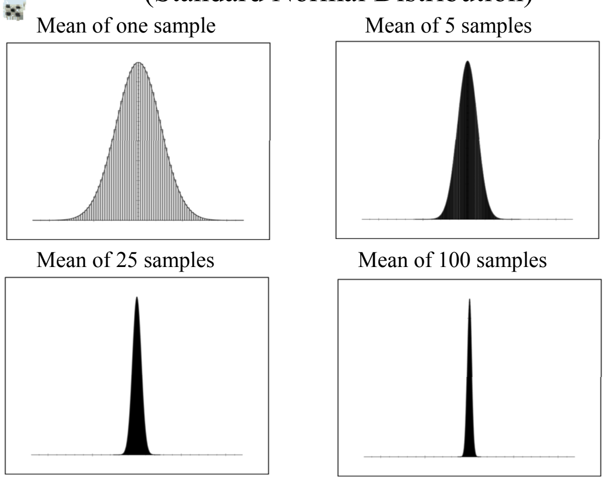
- For most statistical analysis for CS the question is
  - Is my new way better than the old way?
  - Statistically this translates into a statement about the difference between means: “Is the difference between ‘my mean’ and ‘the old mean’ greater than zero?”
- We will approach this question in 2 steps:
  1. What can we say about the true mean of a *single* distribution?
     - Called *point estimation*
  2. How can we compare the true means of *two* or more distributions?

Distribution of the Mean

- Consider the distribution of the average of a set of n independent samples
  - If $n = 1$, the distribution of the average is just the distribution itself, since we have only the single data point
  - If n is larger than one, the distribution of the mean must be narrower than the distribution of the population
    - i.e. the variance and standard deviation must be smaller
  - In fact, the standard deviation of the mean of n samples is given by $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$

Distribution of the Mean

(Standard Normal Distribution)



Confidence Intervals

- As the “finger” gets narrower, the mean of the samples approaches the true mean
- We’d like to say that in the overwhelming majority of all possible experiments, the true mean of this distribution will lie within a specified interval
  - Example: In 99% of cases, the true mean of the distribution, estimated from our 50 samples, lies within the interval [64 , 79] – called a *confidence interval* for the mean

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

t Distribution

- Of course, we don't know the true mean, μ, or true standard deviation, σ
- We *do* know the mean of the samples, $\bar{X}$, the sample size, *n*, and the sample standard deviation, s_X
- If the source distribution is normally distributed, the shape and size of the "finger" is known exactly!
  - We can determine the odds that the true mean lies within a specified range of $\bar{X}$
  - The distribution of the sample average follows a *t* distribution with *n - 1* degrees of freedom, where

$$T = \frac{(\bar{X}-\mu)}{s_{\bar{X}}} = \frac{(\bar{X}-\mu)}{s_X / \sqrt{n}}$$

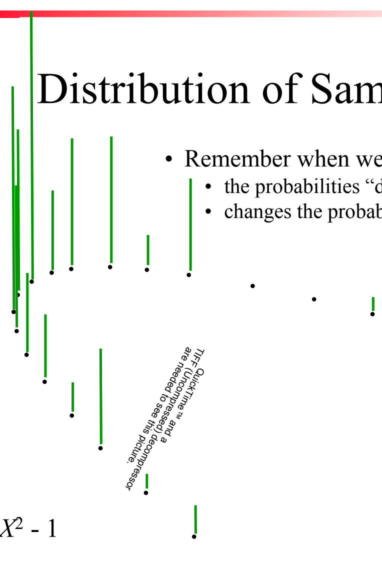
t Distribution

- What is the T random variable's distribution?
- We know that the sample average is normally distributed
  - Sum of normally distributed random variables is normally distributed
  - So numerator is normally distributed
- Standard Deviation based on Variance $\operatorname{var}(X) = E((X-\mu)^2)$
  - the square of a random variable has a different distribution
  - so what is the denominator's probability distribution?

$$T = \frac{(\bar{X}-\mu)}{s_{\bar{X}}} = \frac{(\bar{X}-\mu)}{s_X / \sqrt{n}}$$

Distribution of Sample Variances

- Remember when we square a random variable
  - the probabilities "double-up"
  - changes the probability distribution



$Y = 0.1X^2 - 1$

Chi-Squared Distribution

- Variance has a Chi-Squared Distribution
  - Sample variances have different Chi-Squared distribution
    - Depends on the number of samples
    - Called *degrees of freedom*

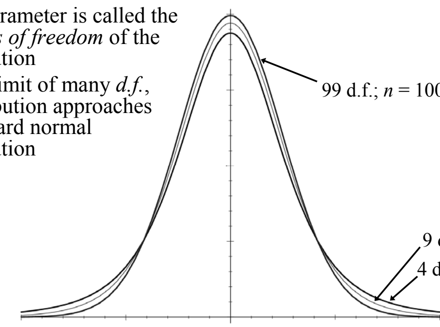
t Distribution

- What is the T random variable's distribution?
- We know that the sample average is normally distributed
  - So numerator is normally distributed
- Standard Deviation, based on Variance
  - so the denominator has a Chi distribution $\mathrm{var}(X) = E((X-\mu)^2)$
- A normal divided by a chi distribution produces a T distribution

$$T = \frac{(\bar{X}-\mu)}{s_{\bar{X}}} = \frac{(\bar{X}-\mu)}{s_X / \sqrt{n}}$$

t Distribution

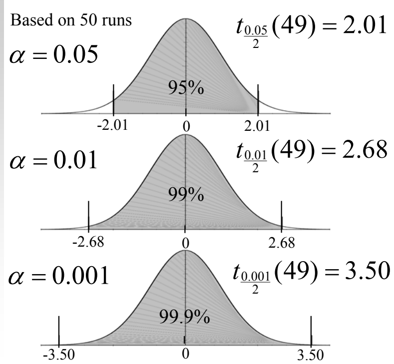
- The *t* "distribution" is really a family of distributions – the shape of the distribution changes as the number of samples, *n*, changes
  - This parameter is called the *degrees of freedom* of the distribution
  - In the limit of many *d.f.*, *t* distribution approaches a standard normal distribution



Estimating the Mean: Confidence Intervals Around the Average

If samples taken from a *standard normal distribution* (μ = 0, σ = 1), the sample average has a *t* distribution.

Based on 50 runs



- For Confidence Intervals, we can use cutoff *t* values
- The wider the cutoff values, the more likely the true mean will fall between them
- α is the probability of obtaining values outside the cutoffs
  - Confidence Level is 1 – α
- Cut off *t* values can be computed using Excel: **=TINV(α, *n* - 1)**
  - Note: TINV() is already 2 sided

Estimating the Mean: Confidence Intervals Around the Average

- We know that

$$T = \frac{(\bar{X}-\mu_X)}{s_X / \sqrt{n}}$$

- Using the $\pm t_{\frac{\alpha}{2}}(n-1)$ cutoff t-values we can form a Confidence Interval that has a 1 - α C.L with *n* - 1 degrees of freedom
- Substituting the cutoff values from the C.I. into the above equation produces

$$\pm t_{\frac{\alpha}{2}}(n-1) = \frac{(\bar{X}-\mu_{\bar{X}})}{s_{\bar{X}} / \sqrt{n}}$$

which can be rewritten as

$$\mu_X = \bar{X} \pm t_{\frac{\alpha}{2}}(n-1)\frac{s_X}{\sqrt{n}}$$

Estimating the Mean: Confidence Intervals Around the Average

- Confidence Intervals can be written in 3 equivalent ways

Error Bounds

$$\mu_X = \overline{X} \pm t_{\frac{\alpha}{2}}(n-1)\frac{s_X}{\sqrt{n}}$$

Confidence Intervals

$$\overline{X} - t_{\frac{\alpha}{2}}(n-1)\frac{s_X}{\sqrt{n}} \le \mu_X \le \overline{X} + t_{\frac{\alpha}{2}}(n-1)\frac{s_X}{\sqrt{n}}$$

$$\mu_X \in \left[\overline{X} - t_{\frac{\alpha}{2}}(n-1)\frac{s_X}{\sqrt{n}}, \overline{X} + t_{\frac{\alpha}{2}}(n-1)\frac{s_X}{\sqrt{n}}\right]$$

Estimating the Mean: Confidence Intervals Around the Average

Example:

- An experimenter runs a New Evolutionary Algorithm on a TSP
- At the end of each run, the smallest length tour that had been found during the run was recorded
- NEA is run 50 times on the same TSP problem
- On average NEA found solutions with a tour length of 272
- The standard deviation of these tours is 87
- We want to compute a Confidence Interval using a 99% Confidence level

Estimating the Mean: Confidence Intervals Around the Average

- From the problem we know that the average NEA run produced tours of

  $\overline{X} = 272$ that had $s_X = 87$

  We know that $\mu_X = \overline{X} \pm t_{\frac{\alpha}{2}}(n-1)\frac{s_X}{\sqrt{n}}$

- Also from the problem $n = 50$ and $\alpha = (1 - 0.99) = 0.01$

  so the $\pm t$ cutoff value is $t_{\frac{0.01}{2}}(50-1) = t_{\frac{0.01}{2}}(49)$

  using Excel we see that TINV(0.01,49) is 2.68

  so $\mu_X = \overline{X} \pm 2.68\frac{s_X}{\sqrt{50}} = \overline{X} \pm 0.38 s_X$

  and so $239 \le \mu_X \le 305$ with a 99% C.L.

i.e. there is only a 1% chance that the true mean lies outside the confidence interval formed around average

Basic Statistical Tests

Comparisons:
Non-Overlapping Confidence Intervals
and the Student's T Test

40

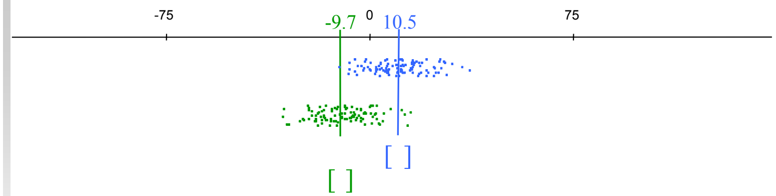
Using Confidence Intervals to Determine Whether My Way is Better

If we have two different EC systems how can we tell if one is better than the other?

Trivial method: Find confidence intervals around both means

- If the CIs don't overlap
  - Then it is a rare occurrence when the two systems do have identical means
  - The system with the better mean can be said to be better on average with a probability better than the Confidence Level
- If the CIs do overlap
  - Can't say that the two systems are different with this technique
  - Either:
    1. The two systems are equivalent
    2. We haven't sampled enough to discriminate between the two

Confidence Interval Example

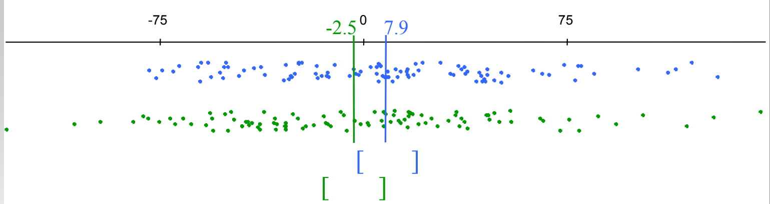


μ	σ
+10	10
-10	10

95% Confidence Level

n	X	s_X	$1.96\frac{s_x}{\sqrt{n}}$	Lower	Upper
100	10.5	10.0	3.3	7.2	13.8
100	-9.7	10.1	3.3	-13.1	-6.4

Confidence Interval Example



μ	σ
+10	50
-10	50

95% Confidence Level

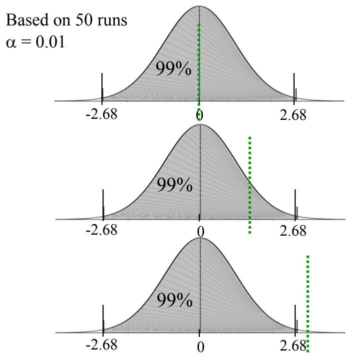
n	X	s_X	$1.96\frac{s_x}{\sqrt{n}}$	Lower	Upper
100	7.9	47.1	9.2	-1.3	17.1
100	-2.5	52.1	10.2	-12.7	7.7

Improving the Sensitivity: The Student *t* Test

- The Student *t* Test is the basic test used in statistics
  - Idea: Gain sensitivity by looking at the difference between the means of the two systems
  - If there is no difference between the actual means of the 2 systems
    - then the difference between the sample averages should be 0, with some error that should follow the *t* distribution
    - this is because the difference btw 2 normal distributions is also normal
    - so the sample average should be a *t* distribution as usual
  - now we can see if the computed difference of the sample averages falls outside a confidence interval (for some α) for the *t* distribution

The Student t Test

Where the normalized difference falls on the t distribution determines whether difference expected if both systems were actually performing the same

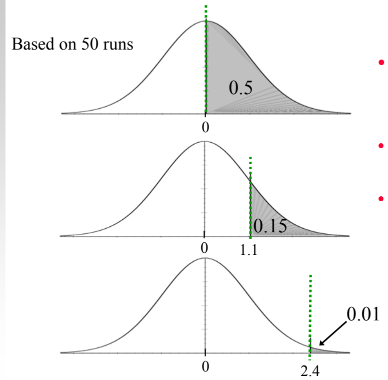


- Normalized difference called the t score

$$t \text{ score} = \frac{\overline{X}_2 - \overline{X}_1}{\sqrt{\frac{s_{X_1}^2}{n_1} + \frac{s_{X_2}^2}{n_2}}}$$

- Distribution again differs for different sample sizes
  - Degrees of Freedom is now $= n_1 + n_2 - 2$
- t test either succeeds or fails
  - t score greater than cutoff for a given C.L. or not

The Student t Test: p-values



- The cut-off values produces a binary decision: true or false
  - loses information
- Better to report the probability that two systems are different
- This is the complement of the probability that they are the same
  - $1 - \Pr(T < t \text{ score})$
  - Called the p-value

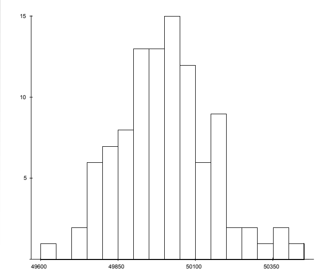
t Test Step by Step

1. Compute the 2 averages X_1 and X_2
2. Compute standard deviations s_1 and s_2
3. Compute degrees of freedom: $n_1 + n_2 - 2$
4. Calculate T statistic: $T = \frac{(\overline{X}_1 - \overline{X}_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$
5. Compute the p-value

- p-value = the area under the t distribution outside [-T, T]
- Use **=TDIST(T, $n_1 + n_2$ - 2, 2)** in Excel
  - The final "2" in Excel means "two-sided"

t Test with Binary Distributions

- Often, we are counting the number of successes versus the number of failures
  - same as counting the number of heads vs number of tails in a coin flip

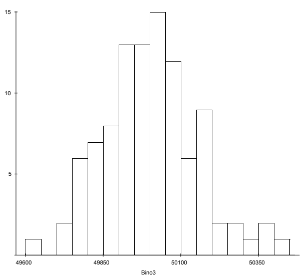


- This produces a Binomial Distribution
  - b is the binomial count for the n repetitions
    - i.e. the number of successes
    - the number of repetitions are called Bernoulli trials
  - p is the true probability of success
    - $q = 1 - p$ is the probability of failure
  - $B \sim B(n, p)$

GECCO'07, *July 7-11, 2007, London, England, United Kingdom.*
ACM 978-1-59593-698-1/07/0007.

t Test with Binary Distributions

- Often, we are counting the number of successes versus the number of failures
  - same as counting the number of heads vs number of tails in a coin flip



- Binomial Distribution
  - $E(b) = np$
  - $Var(b) = np(1-p)$
  - $\sigma_b = \sqrt{np(1-p)}$

t Test with Binary Distributions

- $P = b/n$ is a random variable that equals p as $n \to \infty$
- The sample standard deviation is

$$\sigma_P = \frac{1}{n}\sigma_b = \frac{1}{n}\sqrt{np(1-p)} = \sqrt{\frac{p\cdot(1-p)}{n}} \cong \sqrt{\frac{P\cdot(1-P)}{n}}$$

- The error bounds would be

$$p = P \pm t_{\frac{\alpha}{2}}(n-1)\sqrt{\frac{P(1-P)}{n}}$$

- Two compare two Binomial Distributions, use the *t* Test using the above standard deviation and success frequency

Tests on Non-Normally Distributed Random Variables

Central Limit Theorem

Data Reexpression

Non-Parametric Statistics

51

Assumptions, assumptions

- All we have said so far applies only if the source distribution is a normal distribution
- What if the source distribution is not a normal distribution?
  - In EC, the source distribution is *rarely* normal!
- Fortunately, there is one nice property that can help us out
  - The *Central Limit Theorem*: the sum of many identically distributed random variables tends to a Gaussian
  - Equation of the mean:

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

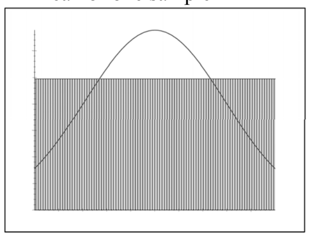
  - So the mean of any set of samples tends to a normal distribution

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

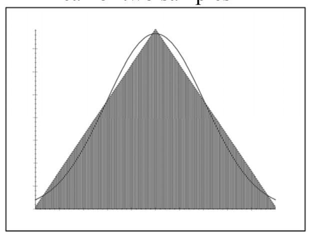
Central Limit Theorem

- The sum of many *independent*, *identically distributed (IID)* random variables approaches a Gaussian normal curve
- E.g. Uniform distribution on [0, 1]:

Mean of one sample



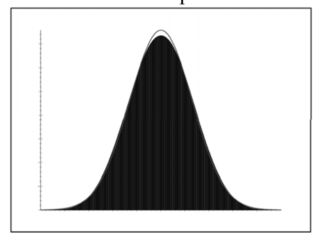
Mean of two samples



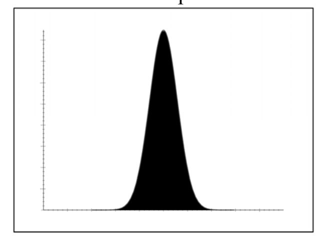
Central Limit Theorem

- E.g. Uniform distribution (continued):

Mean of five samples

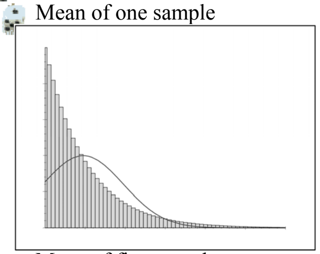


Mean of 25 samples

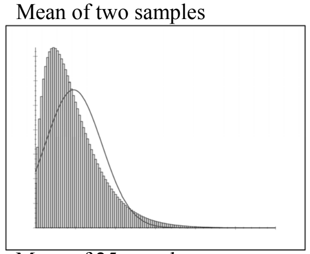


Exponential Distribution

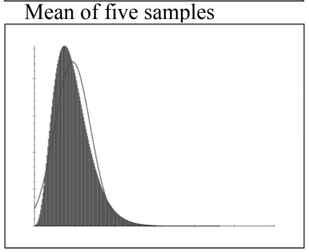
Mean of one sample



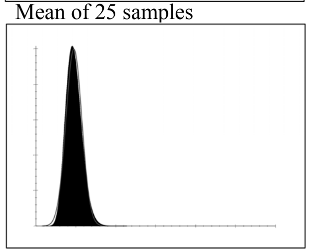
Mean of two samples



Mean of five samples

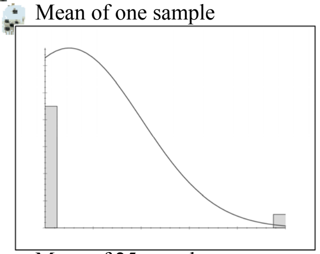


Mean of 25 samples

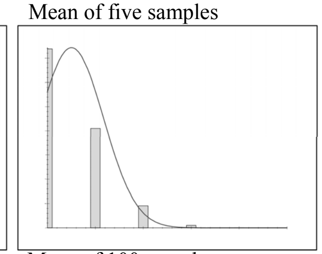


Binomial Distribution (p = 0.1)

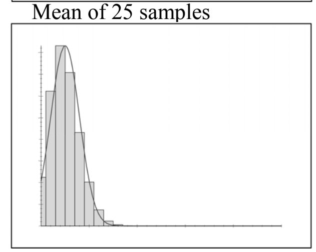
Mean of one sample



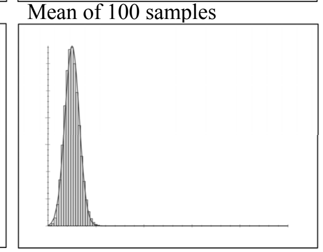
Mean of five samples



Mean of 25 samples



Mean of 100 samples



GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

When The CLT Fails You

- Everything we have done so far depends on the Central Limit Theorem holding
  - But this is not always true
  - ***In in many areas of CS it rarely holds***
- Problems occur when
  - …you have a non-zero probability of obtaining infinity
    - Mean and standard deviation are infinite!
  - …the sample average depends highly on a few scores
    - When the mean of your distribution is not measuring what you want, consider using the median instead (rank-based statistics)
  - …you don't know how fast your sample series converges to normal
    - if your sample average distribution converges very slowly than the number of samples may be *insufficient to assume normality*

So what should we do?

There are 3 techniques:

1. Transforming data to make them normally distributed
   - also called *data re-expression*
   - traditional approach
2. Re-sampling techniques
3. Non-parametric statistics

Data Transformation / Reexpression

- Basic idea
  - transform data so that result is approximately normal
- Reexpression Heuristics

Type	Reexpression Function
Categorical	N/A
Counts	$X \rightarrow \sqrt{X}$
Counted Fractions	Folded power family (see next slide)
Amounts	since X ≥ 0 it is often skewed; then $X \rightarrow \log(X)$
Balances	often difference of two amounts if so transform amounts and take difference or ratio
Bounds	if $X \geq a$ treat $X - a$ as an amount if $a \geq X$ treat $a - X$ as an amount if $a \leq X \leq b$ treat $(X - a)/(b - a)$ as a counted fraction

Data Transformation / Reexpression

- Counted Fractions
  - Bounded from above and below
    - e.g percentages
  - Benefit from reexpressions that stretch their tails
    - Reflects the difficulty of making a counted fraction more extreme as its value approach the edge of the range
    - e.g. Presidential approval rating
      - easy to shift between 55% and 60%,
      - hard to go from 90% to 95%

GECCO'07, *July 7-11, 2007, London, England, United Kingdom.*
ACM 978-1-59593-698-1/07/0007.

Data Transformation / Reexpression

- Counted Fractions
  - Typical reexpressions
    - Plurality $p-(1-p)$
    - Logit $\log(p/(1-p))$
    - Normit/probit/inverse-Gaussian $\mathrm{Gau}^{-1}(p)$
    - Anglit/arc-sine $2\sin^{-1}(\sqrt{p}-\pi/2)$
  - Tukey's lambda family (generalization of all of the above)
    - $\lambda = 1$ plurality
    - $\lambda = 0.5$ folded square
    - $\lambda = 0.41$ anglit (arc sine)
    - $\lambda = 0.14$ probit (inverse Gaussian)
    - $\lambda = 0$ logit

$$\frac{p^{\lambda}-(1-p)^{\lambda}}{\lambda}\frac{1}{2^{\lambda}}$$

trick is finding the right λ

Testing for Normality

- It would be nice to know if a random variable is normally distributed
  - To see if reexpression worked
  - (or if there is no need for remedial measures)
- Many approaches
  - Jarque-Bera test
  - Anderson-Darling test
  - Cramér-von-Mises criterion
  - Lilliefors test for normality
    - Variant of the Kolmogorov-Smirnov (KS) test
  - Pearson's chi-square test
  - Shapiro-Francia test for normality
  - Regression on a normality plot

Testing for Normality: Normality Plot

- Normality plot is a scatter plot
  - Compares with data that one would expect to be produced from a normal distribution
  - If there is a good correlation with your data, then it is normally distributed
    - Scatter plot produces a straight line

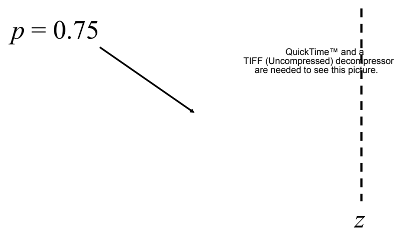
Testing for Normality: Normality Plot

- To create a normality plot
  - Produce known values from a standard normal distribution
    - Generate linear cumulative probabilities
      - $(rank_0 + 0.5)/n$
    - Compute Z-values
      - Use the inverse normal function
        - Takes a probability and produces the Z-value z that 'produces' it when the standard normal curve is integrated from -∞ to z
        - In Excel - NORMSINV(p), where p is a probability
    - We would expect these values to be produced by n samples from a standard normal distribution
    - Called *rankits*

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Testing for Normality: Normality Plot

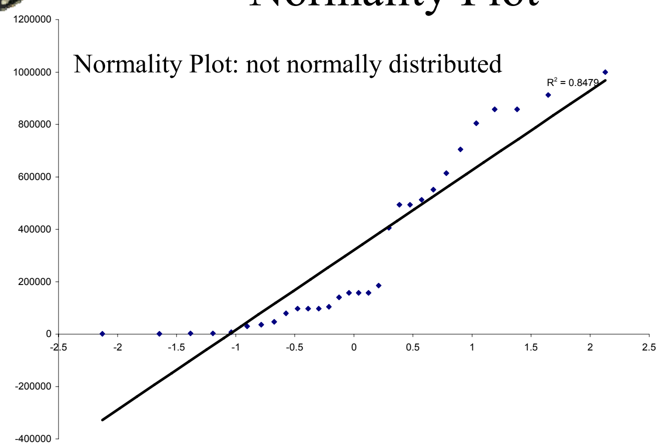
Computing a rankit



Testing for Normality: Normality Plot

- To create a normality plot
  - Sort data
  - Compare sorted data with rankits using a scatter plot
    - Called a *normal probability plot, normality plot,* or *rankit plot*
  - If linear, can assume normal distribution
    - The more linear, the more normal
  - To compute how linear:
    - Add a linear least square regression line to the displayed series
    - Compute r^2
      - a number between 0 (uncorrelated) and 1(linear/correlated)
      - Heuristic: if $r^2 > 0.92$ data can be treated as normally distributed
        if $r^2 > 0.87$ data may be normally distributed
        o.w assume not data is not normally distributed

Testing for Normality: Normality Plot



Resampling

- Estimate the precision of sample statistics (medians, variances, percentiles) by using
  - drawing randomly with replacement from a set of data points
    - Bootstrapping
  - subsets of available data
    - Jackknife
    - Also used in machine learning for training/testing: n-fold validation
- performing significance tests
  - Exchanging labels on data points
    - Permutation test
    - A type of non-parametric statistic

GECCO'07, *July 7-11, 2007, London, England, United Kingdom.*
ACM 978-1-59593-698-1/07/0007.

Non-Parametric Statistics

- Basic Idea
  - Sort the data and then rank them
  - Use Ranks instead of actual values to perform statstics
- Also known as
  - *order statistics*,
  - *ordinal statistics*
  - *rank statistics*
- Measures how interspersed the samples are from the 2 treatments
  - If the result is “alternating” it is assumed that there is no difference
- Can’t be affected by outliers (extrememly large or small values)
  - Just the highest or lowest rank

Non-Parametric Tests

- Reason behind the appropriateness of non-parametric tests
  - Both the sum of ranks and average of ranks will be approximately normally distributed
    - because of the Central Limit Theorem,
    - as long as we have 5 or more samples
  - result is independent of the underlying distribution
- Ranked T-test
  - Perform a *t* test on the ranks of the values
    - instead of the values themselves
- 2 other techniques with similar results are commonly seen
  - Wilcoxon’s Rank-Sum test
  - Mann-Whitney U test
  - All are effectively equivalent

How To Rank the Data

- Augment each data point with a treatment identifier and an additional slot for its rank
- Sort the data sets together by value
  - record the ranks of all values in their rank slot
    - assign the average rank of tied values to each tied value
- Resort by the original order thus splitting the data sets back out
  - keep the combined ranking with each data point
- Apply your *t* test on the ranked values

A	0.03
A	0.91
A	0.64
A	0.99
A	0.64
A	0.16
A	0.16
A	0.91
A	0.16
A	0.27

Two sets of Data

B	0.64
B	0.08
B	0.16
B	0.27
B	0.02
B	0.01
B	0.16
B	0.03
B	0.03
B	0.64

Ranked Example

Ranked Example

A	0.99
A	0.91
A	0.91
A	0.64
A	0.64
B	0.64
B	0.64
A	0.27
B	0.27
A	0.16
A	0.16
A	0.16
B	0.16
B	0.16
B	0.08
A	0.03
B	0.03
B	0.03
B	0.02
B	0.01

Combine the data into a single array and sort

Ranked Example

		ranks
A	0.99	1
A	0.91	2
A	0.91	3
A	0.64	4
A	0.64	5
B	0.64	6
B	0.64	7
A	0.27	8
B	0.27	9
A	0.16	10
A	0.16	11
A	0.16	12
B	0.16	13
B	0.16	14
B	0.08	15
A	0.03	16
B	0.03	17
B	0.03	18
B	0.02	19
B	0.01	20

Give each data element its corresponding rank

Ranked Example

		ranks	
A	0.99	1	
A	0.91	2	t1
A	0.91	3	t1
A	0.64	4	t2
A	0.64	5	t2
B	0.64	6	t2
B	0.64	7	t2
A	0.27	8	t3
B	0.27	9	t3
A	0.16	10	t4
A	0.16	11	t4
A	0.16	12	t4
B	0.16	13	t4
B	0.16	14	t4
B	0.08	15	
A	0.03	16	t5
B	0.03	17	t5
B	0.03	18	t5
B	0.02	19	
B	0.01	20	

Average tied ranks together

t1	2.5
t2	5.5
t3	8.5
t4	12
t5	17

Identify ties

Ranked Example

		ranks	
A	0.99	1	
A	0.91	2.5	t1
A	0.91	2.5	t1
A	0.64	5.5	t2
A	0.64	5.5	t2
B	0.64	5.5	t2
B	0.64	5.5	t2
A	0.27	8.5	t3
B	0.27	8.5	t3
A	0.16	12	t4
A	0.16	12	t4
A	0.16	12	t4
B	0.16	12	t4
B	0.16	12	t4
B	0.08	15	
A	0.03	17	t5
B	0.03	17	t5
B	0.03	17	t5
B	0.02	19	
B	0.01	20	

Average tied ranks together

t1	2.5
t2	5.5
t3	8.5
t4	12
t5	17

Replace tied ranks with average tied ranks

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Ranked Example

		ranks
A	0.99	1
A	0.91	2.5
A	0.91	2.5
A	0.64	5.5
A	0.64	5.5
A	0.27	8.5
A	0.16	12
A	0.16	12
A	0.16	12
A	0.03	17
B	0.64	5.5
B	0.64	5.5
B	0.27	8.5
B	0.16	12
B	0.16	12
B	0.08	15
B	0.03	17
B	0.03	17
B	0.02	19
B	0.01	20

Resort by treatment

Ranked Example

		ranks
A	0.99	1
A	0.91	2.5
A	0.91	2.5
A	0.64	5.5
A	0.64	5.5
A	0.27	8.5
A	0.16	12
A	0.16	12
A	0.16	12
A	0.03	17
B	0.64	5.5
B	0.64	5.5
B	0.27	8.5
B	0.16	12
B	0.16	12
B	0.08	15
B	0.03	17
B	0.03	17
B	0.02	19
B	0.01	20

Perform *t* test on Ranks

	A_{rank}	B_{rank}
avg	7.85	13.15
stdDev	5.28	5.33

	Ranked *t* Test	
$s_T = \sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}}$	2.37	$n = 10$
$(avg_A - avg_B)/s_T$	2.23	t_R score
p-value	0.038	

Ranked *t* Test: What do we pay?

- *t* Test is optimized for the normal distribution
- *t* Test on the ranks is not
  - How much do we pay?

Distribution	# Samples for *t* Test	# Samples for *t* Test on Ranks	# Samples of t_R, normalized to 50 runs of *t*
Normal	31	32	52
Exponential	29	16	27
Uniform	31	34	55
Bimodal	31	34	54
Chubby Tails	40	12	15

A Non-Parametric 'Mean': The Median

- Average of a data set that is not normally distributed produces a value that behaves non-intuitively
  - Especially if the probability distribution is skewed
    - Large values in 'tail' can dominate
    - Average tends to reflect the typical value of the "worst" data not the typical value of the data in general
- Instead use the Median
  - 50^{th} percentile
  - Counting from 1, it is the value in the $\frac{n+1}{2}$ position
    - If *n* is even, $(n+1)/2$ will be between 2 positions, average the values at that position

A Confidence Interval Around the Median: Thompson-Savur

- Find the b the binomial value that has a cumulative upper tail probability of $\alpha/2$
  - b will have a value near $n/2$
- The lower percentile $l = \frac{b}{n-1}$
- The upper percentile $u = 1 - l$
- Confidence Interval is $[value_l, value_u]$
  - i.e. $value_l \le median \le value_u$
  - With a confidence level of $1-\alpha$

A Confidence Interval Around the Median: Thompson-Savur

- In Excel:
  - To calculate b use `CRITBINOM(`n`,`1/2`,`α/2`)`
  - to compute the $value_u$ use the function `PERCENTILE(`dataArray, u`)`
  - to compute the $value_l$ use the function `PERCENTILE(`dataArray, l`)`

A Confidence Interval Alternative to the Ranked *t* Test

- Find the median confidence interval for the two data sets
- If the confidence intervals do not overlap
  - Data sets are taken from different distributions
  - With a confidence level of $1 - \alpha$ where α is the upper tail probability used in computing b
  - Advantages:
    - Gives better understanding of system
      - see median values with error bounds
    - Easy to draw and productive on a graph
  - Disadvantage:
    - Not as sensitive as the ranked *t* test

Effect Size and Repetitions

Cohen's d´

Hedges $\hat{g}$

Number of Repetitions

84

Does My Difference Matter?

- Okay, so your results are significantly better than the published results. So what?
  - Statistics can answer, “is it better?”, but not “does it matter?”
- You perform 100 000 runs of your classifier and 100 000 runs of the reference classifier
  - You get a *t* score of 31.6! ☺
  - The *p*-score is reported by Excel as 0! (Actually 2.0×10^{-219})
  - But…your way classifies data at 91.0% accuracy, whereas the reference technique classifies at 90.8% accuracy.
  - Not much difference!
    - Especially if your technique is much slower than the reference way

Measuring Effect Size

- One statistic for effect size: Cohen’s *d′*
  - *d′* is computed by $d' = \frac{t}{\sqrt{(n_1 + n_2)/2}}$
  - Measures the difference between means in terms of the pooled standard deviation
  - Cohen suggests that 0.25 is a small difference; 0.50 is a medium-sized difference; 0.75 is a large difference
  - For our example, *d′* is 0.10
    - Essentially an insignificant difference
- Problem: we did too many runs!

Hedges’ ĝ

- Problem with Cohen's *d′*
  - d′ is independent if sample sizes
    - Generally good, but there is a problem
  - If one variance is larger than the other
    - the denominator is weighted in that direction
    - the effect size is more conservative
  - But it makes more sense to put stock in the larger sample size
- One solution: Hedges' *ĝ*
  - Hedges and Olkin (1985)
  - Balances respective variances with sample size

$$\hat{g} = \frac{x_1 - x_2}{\sqrt{\frac{(n_1-1)\sigma_1^2 + (n_2-1)\sigma_2^2}{n_1+n_2-2}}} \cdot \left(1 - \frac{3}{4(n_1+n_2)}\right)$$

Perils of Stats for EC

- We can generate lots of data very quickly
  - Leads to over-complicated experimental designs
- Always draw a scatter plot or histogram of your data!
  - This alerts you to strange things
    e.g. the mean is very bad, but some individuals are very good
- Always record the performance of *ALL* the individuals
  - You’ll need this for doing the *t* test on the ranks
  - In EC, we mean ALL individuals of *interest*; i.e. best of run

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Perils of Stats for EC

- Don't confuse Population averages with Best-of-Run averages!
  - In any GA or GP, the average of the population tells you almost nothing of interest
  - Use the median of the best-of-run,
    - do the *WHOLE* experiment several times
  - In GP use the tree size of the best-of-run individuals as well!
    - They are the Heroes – hence they are of interest, unless you're really looking to optimize average tree size during evolution

Repetitions

- What is the number of repetitions needed to see if there is a difference between two means or between two medians?
  - Depends on the underlying distributions
    - But underlying distributions are unknown
- Rule of thumb
  - Perform a minimum of 30 repetitions for each system
  - Performing 50 to 100 repetitions is usually better

Multiple Levels and Factors

Multiple Levels

Post-Hoc Analysis: Bonferonni Correction

Simple Intro to Multiple Factors

Factorial Design

91

More Than 2 Treatments

- Preceding stats to be used for simple experiment designs
- More sophisticated stats needs to be done if:
  - Comparing multiple systems instead of just 2 treatments
    - E.g. comparing the effect on a Genetic Algorithm of using no mutation, low, medium and high levels of mutation
      - We say there are 4 *levels* of the mutation variable
      - Need $\binom{4}{2} = 6$ possible comparisons to test all pairs of treatments
  - Called a 'multi-level' analysis

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Multiple Levels: Post-hoc Analysis

- For 4 levels of mutation there are 6 comparisons possible
  - *Each one* of the comparison holds at a 95% C.L. independent of the other comparisons
  - If *all* comparisons are to hold at once the odds are $0.95 \times 0.95 \times 0.95 \times \ldots \times 0.95 = (0.95)^6 = 0.735$
  - So in practice we only have 73.5% C.L
    - Wrong 1/4 of the time
- For 7 levels of mutation there are 21 comparisons possible
  - C.L. = $(0.95)^{21} = 0.341$
    - Chances are better than half that at least one of the decisions may be wrong!

The Bonferroni Correction for Tests

- To correct, choose a smaller α

$$\alpha' = \frac{\alpha}{m}$$

  - Where m is the number of comparisons
  - So for 95% CL use $\alpha = 0.025/6 = 0.004167$
  - For a Z test the critical value changes from 1.96 to 2.64
- Called a Bonferroni post-hoc correction
  - Other post-hoc techniques such as Tukey and Scheffé that are more powerful than Bonferroni; also Holm’s and Sidak’s procedures can be useful
- You should apply the Bonferroni correction:
  - To t tests (t tests and ranked t tests)
  - To Confidence Intervals and Error Bounds
  - Whenever you mean "all the significant results we found hold at once"

The Bonferroni Correction for Experiments

- The Bonferroni Correction is more widely applicable than just for multi-level comparisons
- We really need to control for the dilution of the confidence levels throughout the study, whether or not the CLs are applied to analyses of independent 'phenomena'
  - We must *divide* the α used for each CL test by the total number of CL tests in the study
- To apply the Bonferroni correction to p-values *multiply* the p-values by the number of CL tests performed
  - “Probabilities” bigger than 1 means “not significant”

The Bonferroni Correction for Experiments

- Example:
  - A robot dog has been created
    - Genetic Programming is used to control the ear wiggles of the robot
    - a Genetic Algorithm is used to optimize its tail wagging ability
  - A study is being done to improve both the ears and the tail independently, and we want to be 95% confident in our over all tests
    - For the ears the GP is tested with 3 different sets of terminal nodes
    - For the tail the GA is tested with 4 different fitness functions
    - There are $\binom{3}{2} + \binom{4}{2} = 3 + 6 = 9$ total CL inferences used in the study
    - Consequently the α used for any CL should be $\alpha = 0.025 / 9 = 0.0028$

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Multiple Factors

- Most of the time, there are many different properties we are interested in studying
  - e.g. We may be trying out various kinds of crossovers, with and without mutation, under different selection pressures
  - Each of the above parameters has multiple levels
  - This is called a multiple factor analysis
    - with each factor having multiple levels
  - Use Analysis of Variance or General Linear Models to analyze
    - See text books on ANOVA and GLMs

Multiple Factors: Factorial Design

- When dealing with multiple factors with multiple levels
  - Important that all combinations of factor levels are tried
  - A given combination of factor levels is called a treatment
  - If you want accurate information about each possible interaction, each treatment should be repeated at least 30 times
    - If you interested largely in main effects, 10 repetitions is often fine, if you have enough levels

Multiple Factors: Factorial Design

E.g. if we have 2 EC systems, new and standard (New and Std) and we want to see their behavior under

- crossover and no crossover (x and ~~x~~)
- 3 different selection pressures (p1, p2 and p3)

	t1	t2	t3	t4	t5	t6	t7	t8	t9	t10	t11	t12
S	New	New	New	New	New	New	Std	Std	Std	Std	Std	Std
X	x	x	x	~~x~~	~~x~~	~~x~~	x	x	x	~~x~~	~~x~~	~~x~~
P	p1	p2	p3	p1	p2	p3	p1	p2	p3	p1	p2	p3

Multiple Factors: Factorial Design

- If we are performing 50 reps per treatment
  - In previous example we have $S \times X \times P \times 50 = 2 \times 2 \times 3 \times 50 = 12 \times 50 = 600$ experiments to perform
- The number of experiments goes up as the product of the number of levels in each factor
  - This is exponential in the number of factors
  - Consequently, carefully choose the factors and factor levels that you study in your experiments
  - Minimize what factors you vary (focus your experiments on the relevant factors)

Statistical Myths

A fun summary…
with some new information

Top 5 Experimental Analysis Myths in CS

i. Results from 1 run is all that is needed
   - No, shows only proof of concept

ii. The best value achieved in a set of runs tells you something about the population distribution
   - No

iii. Using the same random number generator seed for both systems provides a fairer comparison
   - It doesn't - it's the statistical properties of the system that we are looking for

iv. One system is obviously better than the other when looking at the data or graph - no statistics necessary
   - If it is so obvious, then will be easy to show statistically
     - might as well do the stats
     - shows that you are objectively confident in your conclusion

v. "My average is better than yours" means "my technique is better than yours"
   - In the best case you would need to take variance into account

Top 12 Statistics Myths in CS

1. My mean result being better than yours means my technique is superior to yours
   - In the best case you need to perform a *t* test to assert this claim
2. Reporting the mean value of a statistic is good enough
   - You need some representative range
3. Reporting the mean and standard deviation of a statistic is good enough
   - Need number of runs
4. Your data are normally distributed
   - Not usually

Top 12 Statistics Myths in EC

5. The mean performance of the best-of-run individuals of your system is what matters
   - It's usually the median you want
6. 10 runs is enough to show significant differences between groups
   - It can be, but the statistics required to show this are hairy
7. 95% confidence levels are generally sufficient
   - Try 99.9%
8. Drawing 95% confidence intervals around each sample mean on a graph implies that it's a rare event if any of the true means fall outside the CIs
   - Nope; need Bonferroni correction

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.

Top 12 Statistics Myths in EC

9. Reporting the results of several comparisons where each is made at a 95% confidence level means that all conclusions are valid simultaneously
   - Nope; need Bonferroni correction for that too
10. 95% confidence intervals can be computed using the sample mean ± 1.96 standard deviations of the mean
    - Nope; need the Student's *t* score given your degrees of freedom
11. An experimental setup where more than one parameter is varied can be treated like one where exactly one parameter varies
    - Need ANOVA, MANOVA or regression
12. One can infer trends from observed data beyond the data you've generated
    - Generally, this would be a consequence of some model, and you probably haven't supported said model with enough experimental data

References

- Slides online:
  http://www.scs.carleton.ca/~schriste/tamale/UsingAppropriateStatistics.pdf
- Hyperstat Online Textbook:
  - http://davidmlane.com/hyperstat/index.html
  - Statistics textbook for psychology students
    - Easy math, nice examples. ☺
- Statistics Chapter of Numerical Recipes in C
  - http://www.library.cornell.edu/nr/cbookcpdf.html
  - Chapter 14, "Statistical Description of Data"
  - Very detailed, more for advanced users

GECCO'07, July 7-11, 2007, London, England, United Kingdom.
ACM 978-1-59593-698-1/07/0007.