

Optimal Nesting of Species for Exact Cover of Resources:

Two Against Many

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ABSTRACT

The application of resource-defined fitness sharing (RFS) to shape nesting problems reveals a remarkable ability to discover tilings [7, 8]. These tilings represent exact covers for a set of resources, and can be considered a maximally sized set of cooperating (non-competing) species. A recent paper by Horn [9] introduces the first formal analysis of this empirical phenomenon by examining a minimal case: two species a and b “cooperate” to exactly cover the resources, while a third species c “competes” with a and b by overlapping both in terms of covered resources. The analysis reveals that in cases in which a and b *maximally compete* with c for resources, species c will become extinct, while the optimal set of species, a and b , will survive. The current paper generalizes this three-species result by analyzing more complex situations with four or more species. Specifically, we consider two species cooperating against two species competing, and finally two species cooperating against an arbitrary number of competing species. In all cases, proofs are derived that show exactly when the two cooperating species are guaranteed to win out over all competitors. The results are clearly proven using algebra on the niching equilibrium equations for RFS; a purely static analysis.

Categories and Subject Descriptors

I.2.8 [Artificial Intelligence]: Problem Solving, Control Methods, and Search—*Heuristic methods*

General Terms

Algorithms, Theory

Keywords

genetic algorithm, evolutionary computation, coevolution, niching, niches, species, exact cover, tiling, shape nesting

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1. INTRODUCTION

In the 2002 paper [7] that introduced resource-defined fitness sharing (RFS)¹, the niching method exhibits a remarkable ability to converge to an optimal solution on shape nesting problems if that optimal solution consists of a *tiling*. The 2002 paper [7] provides evidence of this phenomenon for both one and two-dimension shape nesting problems. But since RFS operates with quantities defined by sets (e.g., set intersections), and does not make any direct use of the geometric properties of shape nesting, it is possible that this apparent ability to tile one arbitrary shape with another generalizes to an ability to exactly cover a set of resources with a set of subsets of those resources. That is, if an exact cover of the resources (substrate) can be found in the current population, then possibly RFS will always drive the population distribution to represent the exact cover.

It therefore seems important to investigate the ability of RFS to evolve (select) exact covers by applying a theoretical analysis to minimal size cases of tiling (exact cover) problems. A previous paper [9] began such an analysis by examining the minimal case of “cooperation versus competition”: the *two-against-one* case, in which two species cooperate to compete against one other species. In particular, the two cooperative species cover all of the resources, while the third competes for coverage. In this paper we review the analysis in [9] of the two-against-one scenario before generalizing the analysis to cases in which two-species form an exact cover and compete against two or more (arbitrarily many) species.

2. BACKGROUND

We briefly summarize the RFS algorithm and the problem domain of shape nesting, which is a subset of resource covering problems in general. This summary is meant to motivate the analysis at the heart of this paper. We note that RFS was developed as a synthesis of *fitness sharing* (FS) and *resource sharing* (RS). We refer the reader to [7] for more details about the origin of RFS and comparisons with FS and RS.

¹RFS, applied to shape nesting problems, is the subject of U.S. Patent No. 7,181,702.

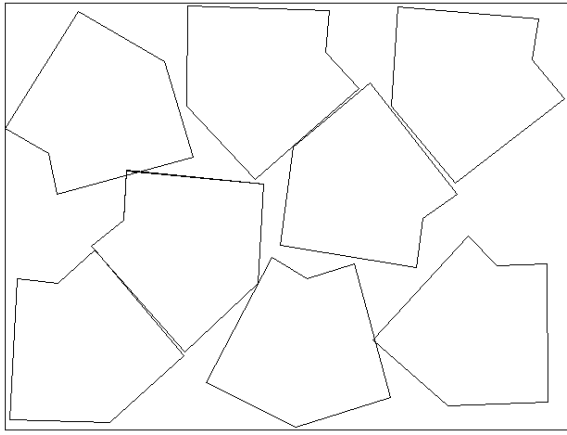


Figure 1: RFS has been shown to nest arbitrary shapes.

2.1 RFS Applied to Shape Nesting

The problem at hand requires “nesting” (that is, placing) shaped pieces on a finite substrate so as to maximize the number of such pieces on the substrate. The objective is often stated, equivalently, as the minimization of “trim” (i.e., unused substrate) [2, 11]. No overlaps among the placed pieces are allowed, and all such pieces must be placed completely within the boundaries of the substrate. Figure 1 illustrates a typical shape nesting problem. The layout of pieces is the result of an actual run.

In [7, 8] the focus is on a very common sub-domain of shape nesting problems: a finite, two-dimensional problem, which means a flat substrate of fixed size, and flat pieces to be nested. The author assumes identical shapes, which means there is only one shape (or “piece”) to be nested. The pieces do not have to be axis-aligned; they can be rotated into any orientation. There are no constraints on the separation or any other relationship between shaped pieces or between the shaped pieces and the substrate boundaries, other than the normal exclusion of overlap.

In the original RFS paper, Horn [7] applies RFS to one and two-dimensional shape nesting problems but limits his tests to axis-aligned squares for the shaped pieces. For example, in the two-dimensional example, illustrated in Figure 2, the task is to nest as many of the smaller square pieces as possible within the larger piece. The width of the substrate square is exactly four times that of the piece square, so that a single optimal solution exists, consisting of sixteen pieces exactly covering the substrate, as shown in Figure 2, right.

In [7], the RFS algorithm is given no information about the solution. The algorithm starts with a random population of 16,000 square pieces (with random positions; the orientation of all pieces are fixed so that all were parallel to the x,y axes). There are 1600 possible piece positions (on a discrete 40 by 40 grid). With a population size of 16,000, the random initial generation contains about 10 copies of each of the 1600 possible species. The GA with RFS is able to select and promote the sixteen *species* corresponding to the solution in Figure 2, right, where each of the 16 species is represented by approximately 1000 copies (individuals) in the final population.

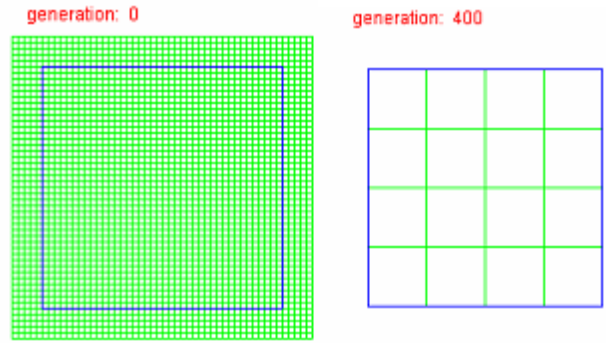


Figure 2: RFS can select for *tilings* of a surface.

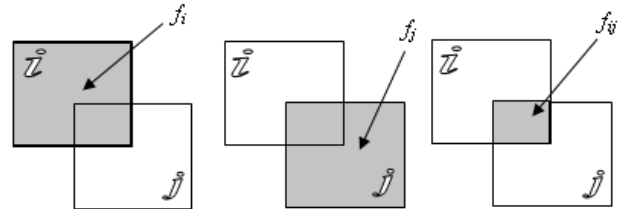


Figure 3: The basic terms used in defining RFS.

2.2 The RFS Algorithm

Under Resource-Defined Fitness Sharing (RFS), every individual of the current population is evaluated and assigned a fitness. In [7], each individual is a chromosome that specifies a placement of a piece. Any individual that extends beyond the boundaries of the substrate is assigned a fitness of 0. All “feasible” individuals (i.e., chromosomes specifying piece placements entirely on the substrate), receive a shared fitness greater than 0, for use in a standard selection method (e.g., tournament selection, proportionate selection).

Under RFS, the shared fitness for each individual is a function of the resources (e.g., area) covered by the individual, and of the extent to which the individual’s coverage overlaps with that of other individuals in the population. The form of the RFS shared fitness formula, $f_{sh,i}$, is that of a fraction:

$$f_{sh,i} = \frac{f_i}{niche_count(i)} = \frac{f_i}{\sum_{j \in P} f_{ij}}, \quad (1)$$

where i is an individual in the population P , f_i is the objective (unshared) fitness of i , and f_{ij} is the pairwise overlap in “coverage” between individuals i and j in P , and $niche_count(i)$ measures the amount of competition for resources covered by i . Under RFS, niche count is defined as the cumulative pairwise overlap between i and other individuals in P . Figure 3 illustrates f_i and f_{ij} for two individuals i and j .

For the purposes of this paper, we normalize the objective fitness f_i to $1 \forall i \in P$. Thus $0 \leq f_{ij} \leq 1, \forall i, j \in P$.

Next we define what we mean by *species* and how the

term relates to individuals. We consider a species to be a set of identical individuals (i.e., with identical coverage of resources). Thus unique chromosomes map one-to-one with unique species. There is complete overlap between any two members of the same species, while there is less than complete overlap between any two members of different species.

Now that we have defined “species”, we can re-write Equation 1 in terms of species:

$$f_{sh,x} = \frac{f_x}{\text{niche_count}(x)} = \frac{f_x}{\sum_{y \in S(P)} n_y f_{xy}}. \quad (2)$$

Equations 2 and 1 are equivalent. Both have an objective fitness in the numerator, and a niche count, calculated over the entire current population, in the denominator. In Equation 1, the summation in the niche count is taken over the population of individuals (using the variable j). In Equation 2, the population is partitioned into a set $S(P)$ of species y , thus $y \in S(P)$. Each species consists of the set of all individuals with the same chromosomes (from the current population). Thus the shared fitness for any member of a species x is equal to the objective fitness of that species divided by the niche count for that species, which is computed as the sum over all species of the interaction term (f_{xy}) multiplied (weighted) by the number of members of that species (i.e., the *species count*: n_y) in the current population P .

2.3 A Static Analysis of Three Species

Horn [9] considers exactly three species, a , b , and c . Their objective (unshared) fitnesses are f_a , f_b , and f_c , representing the amount of resource(s) they cover. There are no other species in the population P . The entire finite population P is divided up among these three species: $S(P) = \{a, b, c\}$. That is, every individual in P is a member of a , b , or c , and all individuals of a species are considered to be identical (at least for the sake of selection). If p_a , p_b , and p_c are the proportions of the population for species a , b , and c respectively, then $0 \leq p_x \leq 1$, $\forall x \in S(P)$, and $p_a + p_b + p_c = 1$. Since we are dealing with proportions, we do not need to name or manipulate an explicit population size $N = |P|$.

2.3.1 RFS Equilibrium

A population distribution is said to be at evolutionary equilibrium if it is equal to the expected distribution of the population after application of the selection operator [12]:

$$E[p_x(t+1)] = p_x(t), \quad (3)$$

where $p_x(t)$ is the proportion of species x in the population at time t , and $E[p_x(t+1)]$ is the expected proportion of x at time $t+1$ (e.g., the subsequent generation). Under RFS and proportionate selection, the expected proportion $p_x(t)$ of species x at time $t+1$ is a function of its proportion $p_x(t)$ at time t and of its shared fitness $f_{sh,x}(t)$ and the population’s average fitness $\bar{f}(t)$ at time t :

$$E[p_x(t+1)] = p_x(t) \frac{f_{sh,x}(t)}{\bar{f}(t)} \quad (4)$$

$$= p_x(t) \frac{f_{sh,x}(t)}{\sum_{y \in S(P)} p_y(t) f_{sh,y}(t)}. \quad (5)$$

Substituting the above expression into Equation 3 and cancelling $p_x(t)$, we find that at equilibrium (here we drop the notation for a specific time step t) the shared fitness $f_{sh,x}$

of all species must be equal to the average fitness,

$$\forall (x \in S(P)) : f_{sh,x} = \sum_{y \in S(P)} p_y f_{sh,y},$$

and therefore must be equal to each other:

$$\forall (x, y \in S(P)) : f_{sh,x} = f_{sh,y}.$$

2.3.2 The General Case

For three species, the most general situation (i.e., arbitrary pair-wise overlaps; no assumptions) has the following equilibrium equations:

$$f_{sh,a} = f_{sh,c}$$

$$f_{sh,c} = f_{sh,b}$$

$$p_a + p_b + p_c = 1.$$

That is, all of the shared fitnesses are the same, so that no individual (a member of a species) and hence no species, receives any preference during selection. Under RFS (as well as under Goldberg and Richardson’s [5] original fitness sharing, FS), the shared fitness of an individual member of species x is simply the share of its objective (static) fitness f_x when divided by the current (dynamic) *niche count* for x :

$$\frac{f_a}{\text{niche_count}(a)} = \frac{f_c}{\text{niche_count}(c)},$$

$$\frac{f_c}{\text{niche_count}(c)} = \frac{f_b}{\text{niche_count}(b)},$$

$$p_a + p_b + p_c = 1.$$

Assuming that all objective fitnesses are equal, then without loss of generality we can normalize them to one, so that $\forall x f_x = 1$. Making this normalization and cross multiplying fractions yields

$$\text{niche_count}(c) = \text{niche_count}(a),$$

$$\text{niche_count}(b) = \text{niche_count}(c),$$

$$p_a + p_b + p_c = 1.$$

Basically, we are setting all niche counts, shown in Equation 2, equal to each other. Next we can substitute the formula for niche count, yielding

$$\sum_{x \in S(P)} (p_x f_{cx}) = \sum_{x \in S(P)} (p_x f_{ax}),$$

$$\sum_{x \in S(P)} (p_x f_{bx}) = \sum_{x \in S(P)} (p_x f_{cx}),$$

$$p_a + p_b + p_c = 1.$$

Expanding the above, we find

$$p_a f_{ac} + p_b f_{bc} + p_c f_{cc} = p_a f_{aa} + p_b f_{ab} + p_c f_{ac},$$

$$p_a f_{ab} + p_b f_{bb} + p_c f_{bc} = p_a f_{ac} + p_b f_{bc} + p_c f_{cc},$$

$$p_a + p_b + p_c = 1.$$

Noting that $\forall x f_{xx} = 1$,

$$p_a f_{ac} + p_b f_{bc} + p_c = p_a + p_b f_{ab} + p_c f_{ac}, \quad (6)$$

$$p_a f_{ab} + p_b + p_c f_{bc} = p_a f_{ac} + p_b f_{bc} + p_c, \quad (7)$$

$$p_a + p_b + p_c = 1. \quad (8)$$

These niching equilibrium equations correspond to the most general situation with three niches/species.

Horn [9] considers several ways to specialize the general situation (e.g., species a and b form a cover, but not an exact one). For this paper, we focus on the particular special case summarized below.

2.3.3 Properties I and II: a and b Form an Exact Cover

Horn [9] specializes Equations 6, 7, and 8 to deal with the “exact cover” by two out of three species. An example of this situation is shown in Figure 4.

Property I: *Minimum $a \leftrightarrow b$ Competition*

$$f_{ab} = 0$$

Now we assume Property I, which is the special case in which species a and b do not overlap, so that $f_{ab} = 0$:

$$p_a f_{ac} + p_b f_{bc} + p_c = p_a + p_c f_{ac} \quad (9)$$

$$p_a f_{ac} + p_b f_{bc} + p_c = p_b + p_c f_{bc} \quad (10)$$

$$p_a + p_b + p_c = 1 \quad (11)$$

Re-arranging Equation 9 above,

$$(f_{ac} - 1)p_a + f_{bc}p_b + (1 - f_{ac})p_c = 0. \quad (12)$$

Horn [9] next makes the key assumption of Property II.

Property II: *Maximum $(a, b) \leftrightarrow c$ Competition*

$$f_{ac} + f_{bc} = f_c$$

Under Property II species c is completely covered by a and b , which means (assuming Property I holds as well) that $f_c = f_{ac} + f_{bc} = 1$ and therefore $f_{ac} = 1 - f_{bc}$.

Substituting $1 - f_{bc}$ for f_{ac} in Equation 12 above gives

$$-f_{bc}p_a + f_{bc}p_b + f_{bc}p_c = 0 \quad (13)$$

$$\Rightarrow p_a - p_b - p_c = 0. \quad (14)$$

Similarly we re-arrange Equation 10,

$$f_{ac}p_a + (f_{bc} - 1)p_b + (1 - f_{bc})p_c = 0.$$

And substituting $1 - f_{ac}$ for f_{bc} ,

$$f_{ac}p_a - f_{ac}p_b + f_{ac}p_c = 0 \quad (15)$$

$$\Rightarrow p_a - p_b + p_c = 0. \quad (16)$$

From Equations 14, 16, and Equation 11, we conclude that

$$(p_a, p_b, p_c) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

Thus Horn shows that if an exact cover of resources exists in a population, then under RFS selection only the species representing the exact cover can be expected to resist invasion by other species at niching equilibrium (with other species being driven to extinction), at least for the case of two-niche exact covers and a single (third) species not part of the exact cover.

We next examine how we can generalize these results. We look at more complex situations with more species.

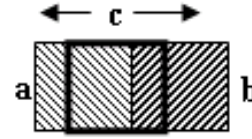


Figure 4: A situation in which Properties I and II hold.

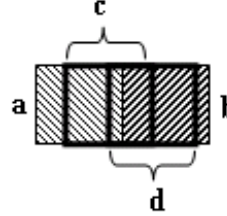


Figure 5: Example of two-against-two, with Properties I and II holding.

3. ANALYSIS: BEYOND THREE SPECIES

Horn [9] covers nearly every niching situation for three niches. We begin our new analysis with four species. We limit our consideration, however, to situations in which two species a and b form the one and only exact cover. Similarly, when we generalize to more than four species, we continue to assume a single exact cover (a and b) only.

3.1 Four Species: Two Against Two

In this situation, we add a fourth species, d , which, like c , is completely covered by a , and b , who together still form an exact cover of all resources. Figure 5 provides an example of this four-species case.

We assume that Property II holds for d as well as for c .

Property II: *Maximum $(a, b) \leftrightarrow d$ Competition*

$$f_{ad} + f_{bd} = f_d$$

In this situation, with Properties I and II holding, the only pairwise overlap that is empty is f_{ab} . The other overlaps are between 0 and 1, exclusive: $0 < f_{ac}, f_{ad}, f_{bc}, f_{bd} < 1$.

Since every species is distinct, if $f_{xy} = 1$ then $x = y$.

3.1.1 At Equilibrium

The niching equilibrium equations for the four species case look like

$$p_a + p_b + p_c + p_d = 1,$$

$$\frac{f_a}{\text{niche_count}(a)} = \frac{f_b}{\text{niche_count}(b)},$$

$$\frac{f_a}{\text{niche_count}(a)} = \frac{f_c}{\text{niche_count}(c)},$$

$$\frac{f_a}{\text{niche_count}(a)} = \frac{f_d}{\text{niche_count}(d)},$$

and after cross-multiplying the fractions and substituting $f_x = 1$:

$$\begin{aligned} p_a + p_b + p_c + p_d &= 1, \\ \text{niche_count}(a) &= \text{niche_count}(b), \\ \text{niche_count}(a) &= \text{niche_count}(c), \\ \text{niche_count}(a) &= \text{niche_count}(d). \end{aligned}$$

Next we substitute the formula for niche count computation:

$$\begin{aligned} p_a + p_b + p_c + p_d &= 1, \\ \sum_{x \in S(P)} (p_x f_{ax}) &= \sum_{x \in S(P)} (p_x f_{bx}), \\ \sum_{x \in S(P)} (p_x f_{ax}) &= \sum_{x \in S(P)} (p_x f_{cx}), \\ \sum_{x \in S(P)} (p_x f_{ax}) &= \sum_{x \in S(P)} (p_x f_{dx}), \end{aligned}$$

then expand,

$$\begin{aligned} p_a + p_b + p_c + p_d &= 1, \\ p_a f_{aa} + p_b f_{ab} + p_c f_{ac} + p_d f_{ad} &= p_a f_{ab} + p_b f_{bb} + p_c f_{bc} + p_d f_{bd}, \\ p_a f_{aa} + p_b f_{ab} + p_c f_{ac} + p_d f_{ad} &= p_a f_{ac} + p_b f_{bc} + p_c f_{cc} + p_d f_{cd}, \\ p_a f_{aa} + p_b f_{ab} + p_c f_{ac} + p_d f_{ad} &= p_a f_{ad} + p_b f_{bd} + p_c f_{cd} + p_d f_{dd}. \end{aligned}$$

3.1.2 Linear Equations

These equations can be re-written as a system of linear equations:

$$\begin{aligned} (f_{aa} - f_{ab})p_a + (f_{ab} - f_{bb})p_b + (f_{ac} - f_{bc})p_c + (f_{ad} - f_{bd})p_d &= 1 \\ (f_{ab} - f_{ac})p_a + (f_{bb} - f_{bc})p_b + (f_{bc} - f_{cc})p_c + (f_{bd} - f_{cd})p_d &= 0 \\ (f_{ac} - f_{ad})p_a + (f_{bc} - f_{bd})p_b + (f_{cc} - f_{cd})p_c + (f_{cd} - f_{dd})p_d &= 0, \end{aligned}$$

which can be represented in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ (f_{aa} - f_{ab}) & (f_{ab} - f_{bb}) & (f_{ac} - f_{bc}) & (f_{ad} - f_{bd}) \\ (f_{ab} - f_{ac}) & (f_{bb} - f_{bc}) & (f_{bc} - f_{cc}) & (f_{bd} - f_{cd}) \\ (f_{ac} - f_{ad}) & (f_{bc} - f_{bd}) & (f_{cc} - f_{cd}) & (f_{cd} - f_{dd}) \end{bmatrix} \begin{bmatrix} p_a \\ p_b \\ p_c \\ p_d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (17)$$

This is a system of four linear equations in four unknowns (p_a, p_b, p_c, p_d) , which can be solved for a unique solution:

$$(p_a, p_b, p_c, p_d) = \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right).$$

As in the three-species scenario, a and b , forming an exact cover, are the only expected survivors at niching equilibrium. If the team of a and b can persevere against one competitor, and against two competitors, perhaps it can “defeat” any number of competitors.

3.2 Many Species: Two Against k

We now try to generalize our previous results to niching scenarios in which the exact cover team of species a and b compete against an arbitrary number, $k > 0$, of competing (i.e., overlapping) species.

To handle an arbitrary number of competitors, we augment our notation by dropping the use of letters for the competing species (i.e., no species c or d) and instead number the k competing species $1..k$. The “exact-cover species” will still be labeled a and b . Thus there are now $k + 2$ distinct species in total, and the non-zero pairwise overlaps are now denoted $f_{a1}, f_{a2}, \dots, f_{ak}$ and $f_{b1}, f_{b2}, \dots, f_{bk}$.

As before, $f_{ab} = 0$ (from Property I), while Property II now implies $\forall i \in 1, \dots, k : f_{ai} + f_{bi} = 1$. As always, $f_{ii} = 1$.

Before proceeding to the equilibrium equations, we organize our algebra by using matrices to represent our system of equations.

3.2.1 Interaction Matrices

First we note that we can organize the pairwise overlaps into a *species interaction matrix*, common in the field of theoretical ecology [13], and similar to (although not the same as) the payoff matrices in game theory [12].

$$\mathbf{M}_{RFS} = \begin{bmatrix} f_{aa} & f_{ab} & f_{a1} & f_{a2} & \dots & f_{ak} \\ f_{ab} & f_{bb} & f_{b1} & f_{b2} & \dots & f_{bk} \\ f_{a1} & f_{b1} & f_{11} & f_{12} & \dots & f_{1k} \\ f_{a2} & f_{b2} & f_{12} & f_{22} & \dots & f_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{ak} & f_{bk} & f_{1k} & f_{2k} & \dots & f_{kk} \end{bmatrix}$$

We note that this square matrix is symmetric about the main diagonal, since niche overlap is a symmetric relationship: $f_{ij} = f_{ji}$. We therefore choose to always write the species subscripts in increasing alphanumeric order, below. We further note that the entries on the main diagonal are all 1, since $\forall i : f_{ii} = 1$. And we recall Property I: $f_{ab} = f_{ba} = 0$. Thus our matrix \mathbf{M}_{RFS} can be re-written:

$$\mathbf{M}_{RFS} = \begin{bmatrix} 1 & 0 & f_{a1} & f_{a2} & \dots & f_{ak} \\ 0 & 1 & f_{b1} & f_{b2} & \dots & f_{bk} \\ f_{a1} & f_{b1} & 1 & f_{12} & \dots & f_{1k} \\ f_{a2} & f_{b2} & f_{12} & 1 & \dots & f_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{ak} & f_{bk} & f_{1k} & f_{2k} & \dots & 1 \end{bmatrix}.$$

It is interesting to consider a partition of this matrix:

$$\mathbf{M}_{RFS} = \begin{bmatrix} 1 & 0 & f_{a1} & f_{a2} & \dots & f_{ak} \\ 0 & 1 & f_{b1} & f_{b2} & \dots & f_{bk} \\ \hline f_{a1} & f_{b1} & 1 & f_{12} & \dots & f_{1k} \\ f_{a2} & f_{b2} & f_{12} & 1 & \dots & f_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{ak} & f_{bk} & f_{1k} & f_{2k} & \dots & 1 \end{bmatrix}.$$

The upper left submatrix consists of all interactions solely among the exact-cover species (a and b). This sub-matrix is an identity matrix, thanks to Property I.

The upper right and lower left sub-matrices represent all interactions solely between the exact-cover species a and b and the competing species ($1..k$). They are transposes of each other. Furthermore, because of Property II, the sum of each column (row) in the upper right (lower left) submatrix is equal to one (that is, $\forall_{i \in (1..k)} f_{ai} + f_{bi} = 1$).

The lower right submatrix contains all interactions solely among the competing species ($1..k$). There are no apparent implications of Properties I or II on this submatrix, because the interactions there are independent of a and b .

3.2.2 Niching Equilibrium Matrix

We can derive a single matrix representing the niching equilibrium equations through matrix operations on \mathbf{M}_{RFS} . (Under RFS, the matrix \mathbf{M}_{RFS} essentially “defines” the niche counts.)

First we note that at niching equilibrium,

$$\begin{bmatrix} 1 & 0 & f_{a1} & f_{a2} & \dots & f_{ak} \\ 0 & 1 & f_{b1} & f_{b2} & \dots & f_{bk} \\ f_{a1} & f_{b1} & 1 & f_{12} & \dots & f_{1k} \\ f_{a2} & f_{b2} & f_{12} & 1 & \dots & f_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{ak} & f_{bk} & f_{1k} & f_{2k} & \dots & 1 \end{bmatrix} \begin{bmatrix} p_a \\ p_b \\ p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix} = \begin{bmatrix} C \\ C \\ C \\ C \\ \vdots \\ C \end{bmatrix},$$

where C is some constant. That is, at niching equilibrium, the niche count for every species is the same ($= C$).

To subtract one niche count from another, we can subtract each row in \mathbf{M}_{RFS} from the row above it. We can do this by pre-multiplying \mathbf{M}_{RFS} by the following matrix \mathbf{M}_{pre} :

$$\mathbf{M}_{pre} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \dots & -1 \end{bmatrix} \quad (18)$$

to get the following matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & f_{a1} - f_{b1} & f_{a2} - f_{b2} & \dots & f_{ak} - f_{bk} \\ 1 - f_{a1} & -f_{b1} & f_{a1} - 1 & f_{a2} - f_{12} & \dots & f_{ak} - f_{1k} \\ 1 - f_{a2} & -f_{b2} & f_{a1} - f_{12} & f_{a2} - 1 & \dots & f_{ak} - f_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 - f_{ak} & -f_{bk} & f_{a1} - f_{1k} & f_{a2} - f_{2k} & \dots & f_{ak} - 1 \end{bmatrix}.$$

We note that the top row has been replaced by all zeros. This is because the niching equilibrium equations that set each one of the $k + 2$ species' niche counts equal to another yields only $k + 1$ independent equations. We therefore need to add the *single population* equation: $\sum_x p_x = 1$. We can now add that to the top row of the matrix above by adding the following matrix \mathbf{M}_{sum} :

$$\mathbf{M}_{sum} = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \quad (19)$$

to get the niching equilibrium matrix \mathbf{M}_{eq} :

$$\mathbf{M}_{pre} \cdot \mathbf{M}_{RFS} + \mathbf{M}_{sum} = \mathbf{M}_{eq} =$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & -1 & f_{a1} - f_{b1} & f_{a2} - f_{b2} & \dots & f_{ak} - f_{bk} \\ 1 - f_{a1} & -f_{b1} & f_{a1} - 1 & f_{a2} - f_{12} & \dots & f_{ak} - f_{1k} \\ 1 - f_{a2} & -f_{b2} & f_{a1} - f_{12} & f_{a2} - 1 & \dots & f_{ak} - f_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 - f_{ak} & -f_{bk} & f_{a1} - f_{1k} & f_{a2} - f_{2k} & \dots & f_{ak} - 1 \end{bmatrix}.$$

The complete system of $k + 2$ equations in $k + 2$ unknowns can then be written:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & -1 & f_{a1} - f_{b1} & f_{a2} - f_{b2} & \dots & f_{ak} - f_{bk} \\ 1 - f_{a1} & -f_{b1} & f_{a1} - 1 & f_{a2} - f_{12} & \dots & f_{ak} - f_{1k} \\ 1 - f_{a2} & -f_{b2} & f_{a1} - f_{12} & f_{a2} - 1 & \dots & f_{ak} - f_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 - f_{ak} & -f_{bk} & f_{a1} - f_{1k} & f_{a2} - f_{2k} & \dots & f_{ak} - 1 \end{bmatrix} \begin{bmatrix} p_a \\ p_b \\ p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

3.2.3 Solving the Niching Equilibrium Equations

We proceed to solve the above system of linear equations for the general case of k competitors, using Gauss-Jordan elimination.

First we note that by Property II, $f_{ax} + f_{bx} = 1$ for $x = 1..k$. So we substitute $1 - f_{bx}$ for f_{ax} where helpful for brevity in \mathbf{M}_{eq} :

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & -1 & 1 - 2f_{b1} & 1 - 2f_{b2} & \dots & 1 - 2f_{bk} \\ f_{b1} & -f_{b1} & -f_{b1} & f_{a2} - f_{12} & \dots & f_{ak} - f_{1k} \\ f_{b2} & -f_{b2} & f_{a1} - f_{12} & f_{a2} - 1 & \dots & f_{ak} - f_{2k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{bk} & -f_{bk} & f_{a1} - f_{1k} & f_{a2} - f_{2k} & \dots & -f_{bk} \end{bmatrix}.$$

Second, to save space while depicting the following algebraic manipulation, we switch to a more concise representation of \mathbf{M}_{eq} , and furthermore only show \mathbf{M}_{eq} and the right-hand vector in each of the following steps. Thus we start with

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & -1 & 1 - 2f_{b1} & \dots & 1 - 2f_{bk} \\ f_{b1} & -f_{b1} & f_{a1} - 1 & \dots & f_{ak} - f_{1k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{bk} & -f_{bk} & f_{a1} - f_{1k} & \dots & -f_{bk} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Next, we reduce the lower left triangle to all zeros. We begin by noting that the first row already has a one in the main diagonal. So we proceed to row two, subtracting the

first row from the second:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -2 & -2f_{b1} & \dots & -2f_{bk} \\ f_{b1} & -f_{b1} & -f_{b1} & \dots & f_{ak} - f_{1k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{bk} & -f_{bk} & f_{a1} - f_{1k} & \dots & -f_{bk} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Next, we divide the second row by -2 and also subtract f_{bx} times the first row, from the $(x+2)^{th}$ row, for $x = 1..k$:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -2f_{b1} & -2f_{b1} & \dots & f_{ak} - f_{1k} - f_{b1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -2f_{bk} & f_{a1} - f_{1k} - f_{bk} & \dots & -2f_{bk} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -f_{b1} \\ \vdots \\ -f_{bk} \end{bmatrix}.$$

Next, we obtain zeros in the second column, below row 2, by adding $2f_{bx}$ times the second row, to the $(x+2)^{th}$ row, for $x = 1..k$:

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & f_{b1} & \dots & f_{bk} \\ 0 & 0 & 2f_{b1}(f_{b1} - 1) & \dots & f_{ak} - f_{1k} + f_{b1}(1 + 2f_{bk}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & f_{a1} - f_{1k} + f_{bk}(1 + 2f_{b1}) & \dots & 2f_{bk}(f_{bk} - 1) \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

We now partition the matrix that we have derived:

$$\left[\begin{array}{cc|ccc} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & f_{b1} & \dots & f_{bk} \\ \hline 0 & 0 & 2f_{b1}(f_{b1} - 1) & \dots & f_{ak} - f_{1k} + f_{b1}(1 + 2f_{bk}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & f_{a1} - f_{1k} + f_{bk}(1 + 2f_{b1}) & \dots & 2f_{bk}(f_{bk} - 1) \end{array} \right] \begin{bmatrix} \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Looking at the lowerright submatrix above, we can see that it represents a homogenous system of linear equations, since the values on the righthand side are all zero. If the original matrix \mathbf{M}_{eq} is non-singular, then so is this submatrix, and there is a unique solution to it. This solution must be the trivial solution: $p_1 = p_2 = \dots = p_k = 0$. Substituting this solution into the first and second rows of our matrix above, we find that $p_a = p_b = \frac{1}{2}$.

THEOREM 1. *If two distinct species \mathbf{a} and \mathbf{b} exactly cover the resources, then under RFS with proportionate selection and an infinite population, a sufficient condition for \mathbf{a} and \mathbf{b} to take over the population, resisting invasion by all competitor species, is the non-singularity of the niching equilibrium matrix.*

PROOF: The proof is given above.

4. DISCUSSION

The RFS algorithm seems unusual among co-evolutionary systems because it lends itself to a static analysis. RFS appears to be amenable to static analysis in part because interaction among individuals, and thus between species, is

limited to pair-wise competition for resources, and because the expression for the niche-count calculation leads to linear equations to describe niching equilibrium. These linear equations can be manipulated for analysis via simple algebra.

4.1 Conclusion

One major result of our analysis is the conclusion that if two species together exactly cover the resources of any and all other species, and if these two species form the only exact cover, then these two species will take over the population at niching equilibrium, resisting “invasion” by any and all other “redundant” species.

One might interpret the relationship between two covering species to be “cooperative” in that they both “compete” against (overlap) k common competitors: additional, and losing, species that are completely covered by the combination of a and b . This is a remarkable result. $k + 2$ species can all have the same *objective* fitness (that is, unshared fitness); they all cover the same amount of resources. Yet RFS selection strongly favors two against the other k . This preference must be due solely to the greater resource coverage of one particular ensemble of species over any other.

We note that the RFS algorithm analysed here, and the analysis itself, are general to all types of RFS applications. The results of this paper are not limited to spatial “nesting” of geometric shapes. The RFS algorithm, and hence the current analysis of the algorithm, apply to the nesting of any kind of set. Shape overlap is really just a special case of set intersection. Thus the most general problem domain to which this analysis is applicable is *exact k-cover* (from the general domain of *set covering*)².

4.2 Limitations

The limitations of this conclusion must be considered. These limitations arise from the assumptions made in the analysis. For example, the use of proportions instead of actual numbers, in keeping track of species representation, means we are using the “infinite population” model, in which we assume that the population is large enough to exactly realize any proportions generated by the manipulation of the equations in our model. More practically, however, populations are generally finite. While we expect that finite populations are well-modeled by the infinite population assumption in many cases [3], a clearer picture of the fidelity of our static models would require some dynamic modeling. For example, we could apply evolutionary game theory analysis to our equilibrium proportions to see if they constitute dynamically stable fixed points [12, 4].

Another way to verify our models (by checking the validity of their assumptions) is through experimentation. While the empirical results in [7, 8], and seen in Figure 2 for example, provide some indication that our models can correctly predict the dynamic behavior of RFS, the actual implementations of RFS in these earlier experiments utilize a form of *tournament selection* rather than the proportionate selection assumed in our model here. New experiments, however, are showing that RFS, under proportionate selection and with a population size of 1000, does indeed promote the exact cover over a third, covered, species (unpublished results).

²Here the integer value k is the fixed size of the subsets used in the cover, and therefore implements our assumption of identical shapes/areas in RFS.

Another limitation of our model is the assumption that only a single, exact cover exists. What about “ties”? Our model does not preclude them, only our assumptions do. For example, in the four niche scenario, the species pair (c,d) , could form an alternative exact cover. We would only have to allow $f_{cd} = 0$, $f_{ac} + f_{ad} = 1$, and $f_{bc} + f_{bd} = 1$. Would RFS “choose” one of the two exact covers? Our model shows no preference. Both covers should be treated equally, with an infinite number of solutions to the set of equilibrium equations, implying genetic drift to the extinction of one pair or the other. Situations with multiple exact covers are worthy of further analysis.

One final limitation seems to have the deepest implications: the non-singularity of the niching equilibrium matrix \mathbf{M}_{eq} . Under what conditions is the niching equilibrium matrix non-singular? Does the non-singularity condition translate to meaningful conditions in the physical world of niche overlaps? Would the singularity/non-singularity of the interaction matrix \mathbf{M}_{RFS} imply singularity/non-singularity of the niching equilibrium matrix \mathbf{M}_{eq} ?

While the analysis of the singularity conditions for the niching equilibrium matrix \mathbf{M}_{eq} is beyond the scope of this paper, we can make some early observations. One such observation is that in the case in which one species completely overlaps another, $f_{ij} = 1, i \neq j$, then both \mathbf{M}_{RFS} and \mathbf{M}_{eq} become singular, since species i and j are identical in their niche overlaps with other species and therefore i and j have identical rows in both of the matrices. But such situations can be considered degenerate if we are interested in $k + 2$ distinct species. Another situation that is likely to be sufficient for singularity is that of multiple solutions, that is, a tie for exact cover, as in the case discussed above for a four-species, two-way tie. But there may be more subtle niching situations resulting in a singular matrix and thus limiting our static analysis (although not necessarily the ability of RFS!) to find a single exact cover even when one exists. Defining and studying such situations might yield insights into important niching/covering dynamics of these evolutionary approaches to hard problems.

4.3 Future Work

This paper presents only an early foray into the theoretical analysis of RFS. A logical next step in the analysis would be to consider the case of h -against- k , where $h > 2$. Would the h exact-covering species still emerge as the sole surviving species at niching equilibrium, when competing against an arbitrarily large number k of other, covered, species? If so, then we will have proven that RFS converges to the optimal tiling in shape nesting problems (e.g., [7]), at least under certain conditions (e.g., non-singular niching equilibrium matrix).

Other directions for future work include a comparison with *resource sharing* [10]. Does resource sharing behave similarly to RFS when there exists in the population an exact cover of the resources? (We note that there seems to be little need to conduct a similar analysis of, and comparison with, *fitness sharing* [5] because FS and RFS use the same functional structure in their sharing functions, with the caveat that FS is limited to rotationally symmetric niche shapes.) Finally, beyond static analysis lie many tools of dynamic analysis of niching and sharing methods [1, 6], such as convergence to equilibrium, stability of equilibrium, using expected proportions over time via recurrence relations, and

Markov chains, which have mostly been applied to two-niche scenarios.

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