Convergence Analysis of Quantum-inspired Genetic Algorithms with the Population of a Single Individual

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ABSTRACT

In this paper, the Quantum-inspired Genetic Algorithms with the population of a single individual are formalized by a Markov chain model using a single and the stored best individual. Here, we analyze the convergence property of the Quantum-inspired Genetic Algorithms based on our proposed mathematical model, and with assumption in which its special genetic operation in the generation changes is restricted to a quantum operator; and show by means of the Markov chain analysis that the algorithm with preservation of the best individual in the population and comparison of it with the existing individual, will converge on the global optimal solution.

Categories and Subject Descriptors:

I.2.3 [**Artificial Intelligence**]: Deduction and Theorem Proving – *deduction*

G.3 [**Mathematics of Computing**]: Probability and Statistics – *Markov Processes*

General Terms: Verification

1. INTRODUCTION

An evolutionary computing algorithm called Quantum-inspired Genetic Algorithm (QGA) is proposed by Han and Kim in [1]. In [2], a simplified model is used to show that this algorithm with a single individual only for ONEMAX problem converges on the global optimum. But here, we model the QGA with a single individual as a Markov chain [3] and then based on the concepts presented in [4], we prove the QGA converges to the global optimum and determine the necessities of its convergence.

2. A MATHEMATICAL MODEL FOR QGA

2.1 Transition Probability Matrix

In the our proposed Markov Chain model for Quantum-inspired Genetic Algorithms (QGAs) with a single individual of m, the state of the Markov chain builds upon both a single individual of the population and the best individual is preserved in the generations. So, the state space S is the set of two components; let $i \in S$ be a state of our Markov chain model:

$$i = \left(B_k \,\middle|\, X_l\right) = (b_1 b_2 \cdots b_m \,|\, x_1 x_2 \cdots x_m) , \tag{1}$$

where B_k (the best individual) and X_l (a single individual) are two strings of length m of 0's and 1's, and integers k and l are identified with their binary representation in $\left(b_1 b_2 \cdots b_m\right)$ and $\left(x_1 x_2 \cdots x_m\right)$, respectively and indexing begins with zero.

The transition probability matrix of our proposed Markov chain model, $Pr = \left(pr_{i,j}\right)$, of size $2^{2m} \times 2^{2m}$, is obtained based on the probability of selecting quantum states of the quantum individuals. In order to determine the values $pr_{i,j}$ which are affected by applying the quantum operator (Equation (2)), to the qubit individual, we consider the changes in probability amplitudes of each qubit that are occurred in the first iteration of `while` loop. And define the transition probability matrix, with considering the square of these probability amplitudes.

$$U(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{2}$$

After applying the quantum operator to the initialized qubit individual to update it, the probability amplitudes of each qubit change to $\begin{bmatrix} \cos(\theta)/\sqrt{2} - \sin(\theta)/\sqrt{2} \\ \sin(\theta)/\sqrt{2} + \cos(\theta)/\sqrt{2} \end{bmatrix}$.

Depending on the execution process of the QGA, it is obvious that occurrence of states in which the fitness value of X_l is greater than B_k, is impossible; in this situation, we assign $pr_{i,i} = 1$ and $pr_{i,j} = 0$, for $i \neq j$. Furthermore, for the formation of transition probability matrix, we assume all of the states $i \in S$ have been ordered by their fitness values f_i.

2.2 Modifier Matrix

The Modifier Matrix that we mention it by $Md = \left(md_{i,j}\right)$, is used to modify the best individual of the existing state; actually,

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for state $i = (B_k \mid X_l)$, if $f(X_l) > f(B_k)$, then $md_{i=(B_k \mid X_l),\, j=(B_l \mid X_l)}$ will be equal to 1 and the other entries became 0.

3. CONVERGENCE ANALYSIS OF QGA

Theorem 1: [3] $P = \begin{bmatrix} C & 0 \\ R & T \end{bmatrix}$ is a partitioned stochastic matrix, where $C : k \times k$ is a regular stochastic square matrix and $R, T \neq 0$.

$$P^{\infty} = \lim_{t\to\infty} P^t = \lim_{k\to\infty} \begin{bmatrix} C^t & 0 \\ \sum_{i=0}^{t-1} T^i R C^{t-i} & T^t \end{bmatrix} = \begin{bmatrix} C^{\infty} & 0 \\ R^{\infty} & 0 \end{bmatrix} \tag{3}$$

is a stable stochastic matrix with $P^{\infty} = 1' p^{\infty}$, where $p^{\infty} = p^0 P^{\infty}$ is unique regardless of the initial distribution, and p^{∞} satisfies: $p_i^{\infty} > 0$ for $1 \le i \le k$ and $p_i^{\infty} = 0$ for $k < i \le n$. □

Now, based on the functions of the two matrices mentioned in the previous Section, the transition matrix for QGA becomes

$$Tr = Pr \cdot Md = \begin{bmatrix} \boldsymbol{Q} & \boldsymbol{0} \\ \boldsymbol{U} & \boldsymbol{V} \end{bmatrix}. \tag{4}$$

Here $\boldsymbol{Q}$ is a submatrix of size $2^m \times 2^m$, which is in the form of

$$\boldsymbol{Q} = \begin{bmatrix} \boxed{P_1} & \boxed{P_2} & \cdots & \boxed{P_{2^m}} \end{bmatrix} \tag{5}$$

where $P_i > 0,\ 1 \le i \le 2^m$ is a column vector of length 2^m; and additionally, $\sum_{i=1}^{2^m} P_i = [1]_{2^m \times 1}$. And $\boldsymbol{0}$ is a zero submatrix of size $2^m \times 2^m(2^m - 1)$, $\boldsymbol{U}$ and $\boldsymbol{V}$ are two nonzero sub-matrices of the sizes $2^m(2^m-1) \times 2^m$ and $2^m(2^m-1) \times 2^m(2^m-1)$, respectively.

Clearly, the transition matrix Tr of the QGA, that is using the quantum gate as its special operator, is non-negative.

Lemma 1: The submatrix $\boldsymbol{Q}$ of the transition matrix Tr is regular.

Proof: The submatrix $\boldsymbol{Q}$ of the transition matrix Tr is stochastic and positive. And with considering this fact, that every positive matrix is regular, the proof is completed. □

Theorem 2: [3] Considering the structure and the eigenvalues of submatrix $\boldsymbol{Q}$ of the transition matrix Tr of the QGA, we have $\lim_{t\to\infty} \boldsymbol{Q}^t \to \boldsymbol{Q}$. □

Theorem 3: The QGA with properties is discussed above converges to the global optimum.

Proof: Submatrix $\boldsymbol{Q}$ gathers the transition probabilities for the states containing a globally best individual. Since $\boldsymbol{Q}$ is a regular stochastic matrix (from **Lemma 1**) and $\boldsymbol{U}$ is a nonzero matrix, Theorem 1 guarantees that the probability of staying in any non-globally optimal state converges to zero for $t \to \infty$.

Besides, in order to have a converged QGA, the absolute value of the rotation angle θ should be $\pi/4$; and its sign must be corresponds to the real best solution; the two last columns of Table 1 let us determine the sign and the range of the rotation angle, but only one of them must be considered in QGA procedure. Satisfying these preconditions, it is easily verified that when $t \to \infty$, the rows of $\boldsymbol{Q}^t$ tend to $(1, 0, \ldots, 0)$, and as a result, convergence of the QGA is proved. □

Table 1. The lookup table

x_i	b_i	$f(X) > f(B)$	θ_i	θ_i
0	0	False / True	$= 0$	$= 0$
0	1	False	$0 <$ $\le \pi/4$	$= 0$
0	1	True	$= 0$	$-\pi/4 \le$ $0 <$
1	0	False	$-\pi/4 \le$ $0 <$	$= 0$
1	0	True	$= 0$	$0 <$ $\le \pi/4$
1	1	False / True	$= 0$	$= 0$

4. CONCLUSIONS

This paper, proposed a new Markov chain model to formalize the QGA with the population of a single individual to analyze its convergence. The analysis showed absolute value of the rotation angle has been effect on the speed of convergence, and its sign determines the direction of convergence.

5. REFERENCES

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