Theory of Randomised Search Heuristics in Combinatorial Optimisation

An Algorithmic Point of View

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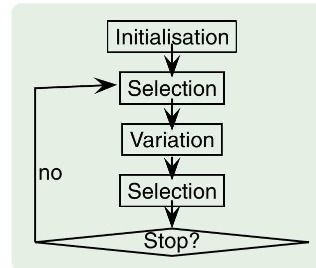
What Are Randomised Search Heuristics (RSHs)?

Most famous example: Evolutionary Algorithms (EAs)

- a bio-inspired heuristic
- paradigm: evolution in nature, "survival of the fittest"



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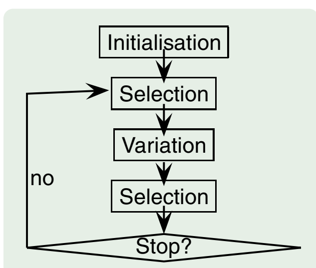

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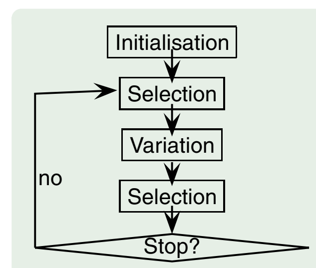
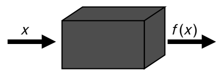


- Goal: optimisation
- Here: discrete search spaces, combinatorial optimisation, in particular pseudo-boolean functions

Optimise $f\colon \{0,1\}^n \to \mathbb{R}$

Why Do We Consider Randomised Search Heuristics?

- Not enough resources (time, money, knowledge) for a tailored algorithm
- Black Box Scenario 

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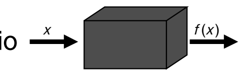
- Black Box Scenario 

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- We like the simplicity, robustness, ... of Randomised Search Heuristics
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My point of view

Do not only consider RSHs empirically. We need a solid theory to understand how (and when) they work.

What RSHs Do We Consider?

Theoretically considered RSHs

- (1+1) EA
- (1+λ) EA (offspring population)
- (μ+1) EA (parent population)
- (μ+1) GA (parent population and crossover)
- GIGA (crossover)
- SEMO (multi-objective)
- Randomised Local Search (RLS)
- Metropolis Algorithm/Simulated Annealing (MA/SA)
- Ant Colony Optimisation (ACO)
- ...

First of all: define the simple ones

The Most Basic RSHs

(1+1) EA, RLS, MA and SA for maximisation problems

(1+1) EA

1. Choose $x_0 \in \{0, 1\}^n$ uniformly at random.
2. For $t := 1, \ldots, \infty$
   1. Create y by flipping each bit of x_t indep. with probab. $1/n$.
   2. If $f(y) \geq f(x_t)$ set $x_{t+1} := y$ else $x_{t+1} := x_t$.

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RLS

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MA

1. Choose $x_0 \in \{0,1\}^n$ uniformly at random.
2. For $t := 1, \dots, \infty$
   1. Create y by flipping one bit of x_t uniformly.
   2. If $f(y) \geq f(x_t)$ set $x_{t+1} := y$
      else $x_{t+1} := y$ with probability $e^{(f(x_t)-f(y))/T}$ anyway
      and $x_{t+1} := x_t$ otherwise.

T is fixed over all iterations.

The Most Basic RSHs

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SA

1. Choose $x_0 \in \{0,1\}^n$ uniformly at random.
2. For $t := 1, \dots, \infty$
   1. Create y by flipping one bit of x_t uniformly.
   2. If $f(y) \geq f(x_t)$ set $x_{t+1} := y$
      else $x_{t+1} := y$ with probability $e^{(f(x_t)-f(y))/T_t}$ anyway
      and $x_{t+1} := x_t$ otherwise.

T_t is dependent on t, typically decreasing

What Kind of Theory Are We Interested In?

- Not interesting here: convergence (often trivial), local progress, models of EAs (e. g., infinite populations), ...
- Treat RSHs as randomised algorithm!
- Analyse their "runtime" on selected problems

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Definition

Let RSH A optimise f. Each f-evaluation is counted as a time step. The *runtime* $T_{A,f}$ of A is the random first point of time such that A has sampled an optimal search point.

- Often considered: expected runtime, distribution of $T_{A,f}$
- Asymptotical results w. r. t. n

How Do We Obtain Results?

We use (rarely in their pure form):

- Coupon Collector's Theorem
- Principle of Deferred Decisions
- Concentration inequalities:
  Markov, Chebyshev, Chernoff, Hoeffding, . . . bounds
- Markov chain theory: waiting times, first hitting times
- Rapidly Mixing Markov Chains
- Random Walks: Gambler's Ruin, drift analysis (Wald's equation), martingale theory, electrical networks
- Random graphs (esp. random trees)
- Identifying typical events and failure events
- Potential functions and amortised analysis
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Adapt tools from the analysis of randomised algorithms; understanding the stochastic process is often the hardest task.

Early Results

Analysis of RSHs already in the 1980s:

- Sasaki/Hajek (1988): SA and Maximum Matchings
- Sorkin (1991): SA vs. MA
- Jerrum (1992): SA and Cliques
- Jerrum/Sorkin (1993, 1998): SA/MA for Graph Bisection
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Since the early 1990s

Systematic approach for the analysis of RSHs, building up a completely new research area

This Tutorial

How the Systematic Research Began — Toy Problems

Simple example functions (test functions)

- $\text{OneMax}(x_1, \dots, x_n) = x_1 + \dots + x_n$
- $\text{LeadingOnes}(x_1, \dots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{i} x_j$
- $\text{BinVal}(x_1, \dots, x_n) = \sum_{i=1}^{n} 2^{n-i} x_i$
- polynomials of fixed degree

Goal: derive first runtime bounds and methods

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Artificially designed functions

- with sometimes really horrible definitions
- but for the first time these allow rigorous statements

Goal: prove benefits and harm of RSH components,
e. g., crossover, mutation strength, population size ...

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Example: OneMax

Theorem (e. g., Droste/Jansen/Wegener, 1998)
The expected runtime of the RLS, (1+1) EA, (μ+1) EA, (1+λ) EA on OneMax *is* $\Omega(n \log n)$.

Proof by modifications of Coupon Collector's Theorem.

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Proof by modifications of Coupon Collector's Theorem.

Theorem (e. g., Mühlenbein, 1992)
The expected runtime of RLS and the (1+1) EA on OneMax *is* $O(n \log n)$.

Holds also for population-based (μ+1) EA and for (1+λ) EA with small populations.

Proof of the $O(n \log n)$ bound

- *Fitness levels:* $L_i := \{x \in \{0,1\}^n \mid |x|_1 = i\}$

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- (1+1) EA never decreases its current fitness level.
- From i to some higher-level set with prob. at least

$$\underbrace{\binom{n-i}{1}}_{\text{choose a 0-bit}} \cdot \underbrace{\left(\frac{1}{n}\right)}_{\text{flip this bit}} \cdot \underbrace{\left(1-\frac{1}{n}\right)^{n-1}}_{\text{keep the other bits}} \geq \frac{n-i}{en}$$

- Expected time to reach a higher-level set is at most $\frac{en}{n-i}$.
- Expected runtime is at most

$$\sum_{i=0}^{n-1} \frac{en}{n-i} = O(n \log n). \qquad \square$$

Later Results Using Example Functions

- Find the theoretically optimal mutation strength ($1/n$ for OneMax!).
- optimal population size (often 1!)
- crossover vs. no crossover → Real Royal Road Functions
- multistarts vs. populations
- frequent restarts vs. long runs
- dynamic schedules
- ...

Later Results Using Example Functions

- Find the theoretically optimal mutation strength ($1/n$ for OneMax!).
- optimal population size (often 1!)
- crossover vs. no crossover → Real Royal Road Functions
- multistarts vs. populations
- frequent restarts vs. long runs
- dynamic schedules
- ...

Further reading: Droste/Jansen/Wegener (2002), He/Yao (2002, 2003), Jansen (2002), Jansen/De Jong/Wegener (2005), Jansen/Wegener (2001, 2005), Storch/Wegener (2004), Witt (2006)

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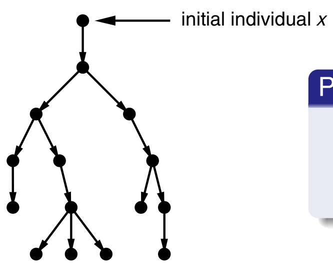
An Advanced Example: (μ+1) EA
Definition

(μ+1) EA

Convention: multisets

1. Choose $P_0 := \{x_1, \ldots, x_\mu \in \{0,1\}^n\}$ uniformly at random.
2. For $t := 1, \ldots, \infty$
   1. Choose x from P_t uniformly at random.
   2. Create y by flipping each bit of x indep. with probab. $1/n$.
   3. Set $P^* := P_t \cup \{y\}$.
   4. Choose x with lowest f-value in P^* uniformly.
   5. Set $P_{t+1} := P^* \setminus \{x\}$.

Advanced Example: (μ+1) EA and Family Trees
Properties of Trees



Properties

- nodes = descendants of x
- new node after each mutation of a descendant

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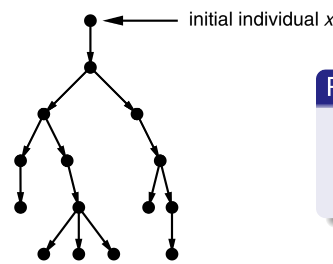

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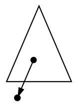
- nodes = descendants of x
- new node after each mutation of a descendant

- Interesting: depth of the tree since low depth $\longrightarrow$ few progress
- What stochastic process creates the tree?

The Process Behind Family Trees

Sequence of trees T_t such that

- at time 0, there is only the root,
- at time t, either nothing happens ($T_{t+1} = T_t$), or



node from T_t is chosen and new leaf appended $\rightarrow T_{t+1}$.

Crucial: each node chosen with prob. at most $1/\mu$.

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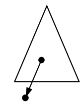

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Technical Lemma (Witt, 2006)

Depth of tree at time t: at most $\frac{3t}{\mu}$ with prob. $1 - 2^{-\Omega(t/\mu)}$.

Proof of Technical Lemma

- Each path has a unique history $t_1, \dots, t_\ell$ s. t. i-th node appears at time t_i.
- Prob(path with history $t_1, \dots, t_\ell$ created) $\le \left(\frac{1}{\mu}\right)^\ell$

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- Prob(path with history $t_1, \dots, t_\ell$ created) $\le \left(\frac{1}{\mu}\right)^\ell$
- Consider at most t steps: at most $\binom{t}{\ell}$ choices for $0 \le t_1 < t_2 < \cdots < t_\ell \le t$.
- Prob($\exists$ path of length ℓ after $\ell\mu/3$ steps)

$$\le \binom{\ell\mu/3}{\ell}\left(\frac{1}{\mu}\right)^\ell \le \left(\frac{e\ell\mu}{3\ell}\right)^\ell \left(\frac{1}{\mu}\right)^\ell = 2^{-\Omega(\ell)}. \qquad \square$$

Application: General Lower Bound

Theorem (Witt, 2006)

Let f be a function with a unique optimum and $\mu = poly(n)$. Then the runtime of the $(\mu{+}1)$ EA on f is $\Omega(\mu n)$ with probability $1 - 2^{-\Omega(n)}$.

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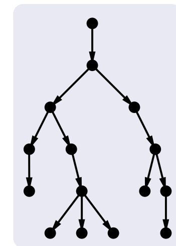
Proof idea:

- W. o. p.: after $\mu n/12$ steps: all paths in family trees have length $\leq n/4$.
- W. o. p.: initially, for all individuals: Hamming distance $\geq n/3$ from optimum.
- W. o. p.: $n/4$ mutations do not overcome Hamming distance $\geq n/3$.



RSHs for Combinatorial Optimisation

- Analyse runtime and approximation quality on well-known combinatorial optimisation problems, e. g.,
  - sorting problems (is this an optimisation problem?),
  - shortest path problems,
  - Eulerian cycles,
  - mininum spanning trees,
  - maximum matchings,
  - partition problem,
  - set cover problem,
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  - ...
- What we do not hope: to be better than the best problem-specific algorithms
- In the following no fine-tuning of the results

This Tutorial

(1+1) EA for the Minimum Spanning Tree Problem

n nodes, m edges: bit string from $\{0, 1\}^m$ selects edges

Fitness function: weight of tree/leading to trees for non-trees

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The expected time until the (1+1) EA has created an MST is bounded by $O(n^4(\log n + \log w_{\max}))$.

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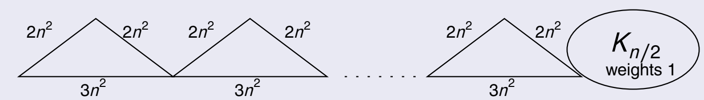
A Tight Example



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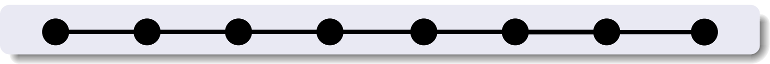


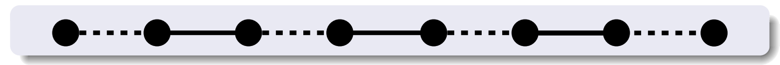
Theorem (Giel/Wegener, 2003)

The expected time until the (1+1) EA finds a maximum matching on a path of n edges is $O(n^4)$.

(1+1) EA for the Maximum Matching Problem
The Behaviour on Paths (2)

Proof idea:

- Consider a second-best matching.
- Is there a free edge? Flip one bit! $\rightarrow$ probability $\Theta(1/n)$.
- Else 2-bit flips $\rightarrow$ probability $\Theta(1/n^2)$.



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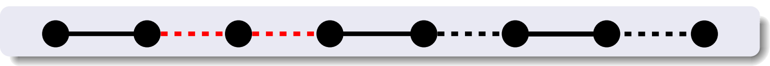

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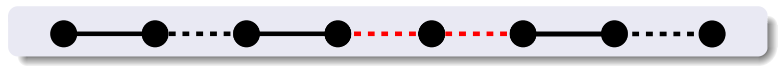

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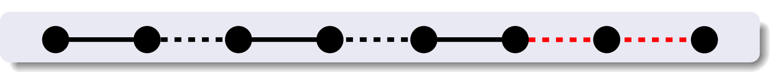
- Then flip the free edge!



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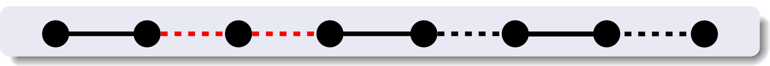

(1+1) EA for the Maximum Matching Problem
The Behaviour on Paths (2)

Proof idea:
- Consider a second-best matching.
- Is there a free edge? Flip one bit! → probability $\Theta(1/n)$.
- Else 2-bit flips → probability $\Theta(1/n^2)$.
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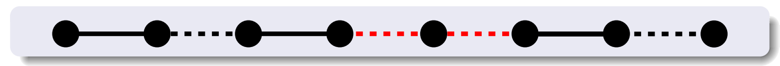

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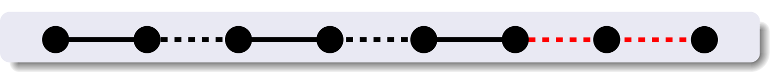

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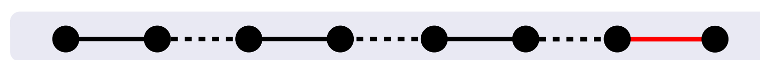

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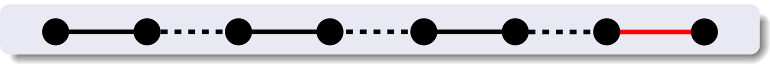
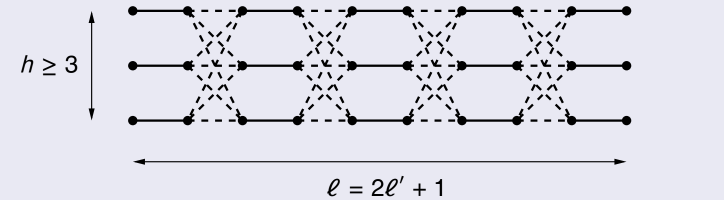


- Length changes according to a fair random walk (Gambler's Ruin Problem) → Expected runtime $O(n^2) \cdot O(n^2) = O(n^4)$.

(1+1) EA for the Maximum Matching Problem
A Negative Result

Worst-case graph (Sasaki/Hajek, 1988)



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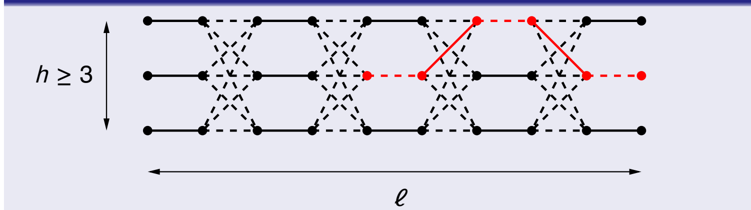

Augmenting path

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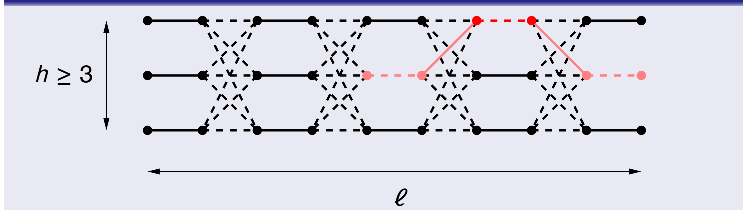

Augmenting path can get shorter

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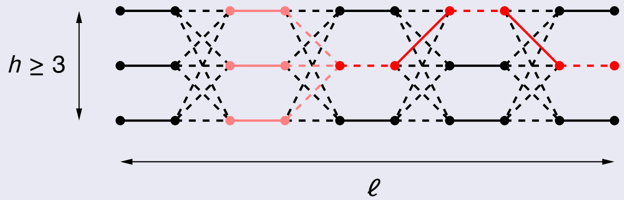

Augmenting path can get shorter but is more likely to get longer.

Theorem

For $h \geq 3$, the (1+1) EA has exponential expected runtime $2^{\Omega(\ell)}$ on $G_{h,\ell}$.

Proof by drift analysis

(1+1) EA for the Maximum Matching Problem
(1+1) EA is a PRAS

Insight: do not hope for exact solutions but for approximations

Theorem (Giel/Wegener, 2003)

For $\varepsilon > 0$, the (1+1) EA finds a $(1+\varepsilon)$-approximation of a maximum matching in expected time $O(m^{2\lceil 1/\varepsilon \rceil})$ and is a polynomial-time randomised approximation scheme (PRAS).

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Proof idea:

- Look into the analysis of the Hopcroft/Karp algorithm.
- Current solution worse than $(1+\varepsilon)$-approximate $\rightarrow$ many augmenting paths, in partic. a short one of length $\leq 2\lceil \varepsilon^{-1} \rceil$
- Wait for the (1+1) EA to optimise this short path.

A More General View

Minimum spanning trees and bipartite matching are special cases of matroid optimisation problems.

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Let E be a finite set and $\mathcal{F} \subseteq 2^E$. $M = (E, \mathcal{F})$ is a *matroid* if

(i) $\emptyset \in \mathcal{F}$,
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(iii) $\forall X, Y \in \mathcal{F}, |X| > |Y| : \exists x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{F}$.

Adding a function $w : E \to \mathbb{N}$ yields a weighted matroid.

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Exemplary Results (Reichel and Skutella, 2007)

The (1+1) EA and RLS solve the matroid optimisation problems

- min. weight basis exactly in time $O(|E|^2(\log |E| + \log w_{\max}))$.
- unweighted intersection up to $1 - \varepsilon$ in time $O(|E|^{2\lceil 1/\varepsilon \rceil})$.

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Very abstract/general, a step towards a characterisation of polynomially solvable problems on which EAs are efficient

This Tutorial

(1+1) EA and the Partition Problem

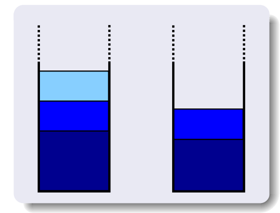
What about NP-hard problems? → Study approximation quality

(1+1) EA and the Partition Problem

What about NP-hard problems? → Study approximation quality

For $w_1, \dots, w_n$, find $I \subseteq \{1, \dots, n\}$ minimising

$$\max\left\{\sum_{i\in I} w_i, \sum_{i\notin I} w_i\right\}.$$



(1+1) EA and the Partition Problem

What about NP-hard problems? → Study approximation quality

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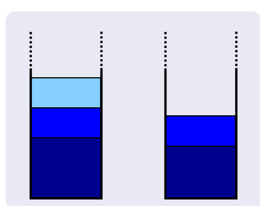

This is an "easy" NP-hard problem:

- not strongly NP-hard,
- FPTAS exist,
- ...

(1+1) EA for the Partition Problem
Worst-Case Results

Coding: bit string $\{0,1\}^n$ characteristic vector of I

Fitness function: weight of fuller bin

Theorem (Witt, 2005)

On any instance for the partition problem, the (1+1) EA reaches a solution with approximation ratio 4/3 in expected time $O(n^2)$.

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Theorem (Witt, 2005)

There is an instance such that the (1+1) EA needs with prob. $\Omega(1)$ at least $n^{\Omega(n)}$ steps to find a solution with a better ratio than $4/3-\varepsilon$.

Proof ideas: study effect of local steps and local optima

(1+1) EA for the Partition Problem
Worst Case – PRAS by Parallelism

Theorem (Witt, 2005)

On any instance, the (1+1) EA with prob. $\geq 2^{-c\lceil 1/\varepsilon\rceil \ln(1/\varepsilon)}$ finds a $(1+\varepsilon)$-approximation within $O(n\ln(1/\varepsilon))$ steps.

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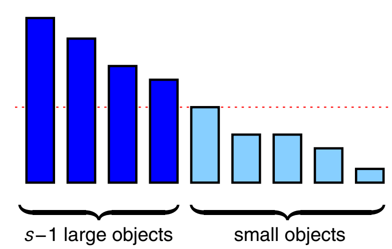
- $2^{O(\lceil 1/\varepsilon\rceil \ln(1/\varepsilon))}$ parallel runs find a $(1+\varepsilon)$-approximation with prob. $\geq 3/4$ in $O(n\ln(1/\varepsilon))$ parallel steps.
- Parallel runs form a PRAS!

(1+1) EA for the Partition Problem
Worst Case – PRAS by Parallelism (Proof Idea)

Set $s := \lceil \frac{2}{\varepsilon} \rceil$ and $w := \sum_{i=1}^{n} w_i$.

Assuming $w_1 \geq \cdots \geq w_n$, we have $w_i \leq \varepsilon \frac{w}{2}$ for $i \geq s$.



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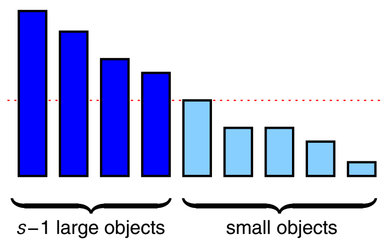

Analyse probability of distributing

- large objects in an optimal way,
- small objects greedily $\Rightarrow$ additive error $\leq \varepsilon w/2$,

This is the algorithmic idea by Graham (1969).

(1+1) EA for the Partition Problem
Average-Case Analyses

Models: each weight drawn independently at random, namely

1. uniformly from the interval $[0, 1]$,
2. exponentially distributed with parameter 1 (i. e., $\text{Prob}(X \geq t) = e^{-t}$ for $t \geq 0$).

Approximation ratio no longer meaningful, we investigate:
discrepancy = absolute difference between weights of bins.

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How close to discrepancy 0 do we come?

(1+1) EA for the Partition Problem
Partition Problem - Known Averge-Case Results

Deterministic, problem-specific heuristic LPT

Sort weights decreasingly,
put every object into currently emptier bin.

Analysis in both random models:
After LPT has been run, additive error is $O((\log n)/n)$
(Frenk/Rinnooy Kan, 1986).

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Can RLS or the (1+1) EA reach a discrepancy of $o(1)$?

(1+1) EA for the Partition Problem
New Result

Theorem (Witt, 2005)

In both models, the (1+1) EA reaches discrepancy $O((\log n)/n)$ after $O(n^{c+4} \log^2 n)$ steps with probability $1 - O(1/n^c)$.

Almost the same result as for LPT!

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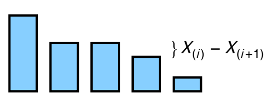
Proof exploits order statistics:

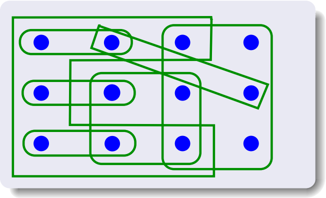
W. h. p.
$X_{(i)} - X_{(i+1)} = O((\log n)/n)$
for $i = \Omega(n)$.



This Tutorial

The Set Cover Problem

Another NP-hard problem



Given:

- ground set S,
- collection $C_1, \ldots, C_n$ of subsets with positive costs $c_1, \ldots, c_n$.

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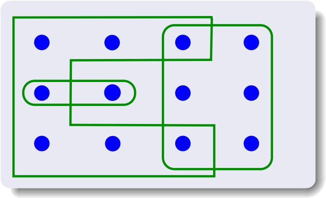

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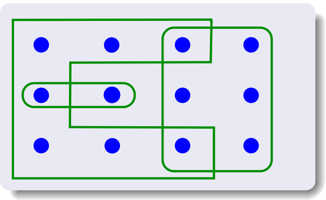

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Traditional single-objective approach

Fitness = cost of selection of subsets, penalty for non-covers

Theorem

There is a Set Cover instance parameterized by $c > 0$ such that RLS and the (1+1) EA for any c need an infinite resp. exponential expected time to obtain a c-approximation.

Multi-objective Optimisation

Fitness $f\colon \{0,1\}^n \to \mathbb{R} \times \mathbb{R}$ has two objectives:

1. minimise the cost of the selection,
2. minimise the number of uncovered elements from S.

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Simple Evolutionary Multi-objective Optimiser (SEMO)

1. Choose $x \in \{0,1\}^n$ uniformly at random.
2. Determine $f(x)$.
3. $P \leftarrow \{x\}$.
4. Repeat
   - Choose $x \in P$ uniformly at random.
   - Create x' by flipping one randomly chosen bit of x.
   - Determine $f(x')$.
   - If x' is not dominated by any other search point in P, include x' into P and delete all other solutions $z \in P$ with $f(x') \le f(z)$ from P.

Achieving Almost Best-possible Approximations

Theorem (Friedrich, He, Hebbinghaus, Neumann, Witt, 2007)

For any instance of the Set Cover problem, SEMO finds an $(\ln|S|+1)$*-approximate solution in expected time* $O(n|S|^2 + n|S|(\log n + \log c_{\max}))$.

Proof idea:

- Greedy procedure by cost-effectiveness: stepwise choose sets covering new elements at minimum average cost.

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- Potential k: SEMO covers k elements at cost $\le \sum_{i=k+1}^{|S|} \frac{\mathrm{OPT}}{i}$.

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It probably cannot be done better in polynomial time.

This Tutorial

1. The origins: example functions and toy problems
   - A simple toy problem: OneMax for (1+1) EA
   - An advanced example: population size for the (μ+1) EA
2. Combinatorial optimisation problems
   - (1+1) EA and minimum spanning trees
   - (1+1) EA and maximum matchings
   - (1+1) EA and the partition problem
   - Multi-objective optimisation and the set cover problem
   - SA beats MA in combinatorial optimisation
   - ACO and minimum spanning trees
3. End
4. References

Simulated Annealing Beats Metropolis in Combinatorial Optimisation

Jerrum/Sinclair (1996)

"It remains an outstanding open problem to exhibit a natural example in which simulated annealing with any non-trivial cooling schedule provably outperforms the Metropolis algorithm at a carefully chosen fixed value" of the temperature.

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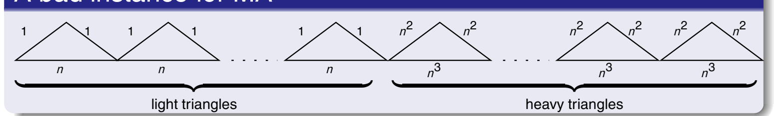
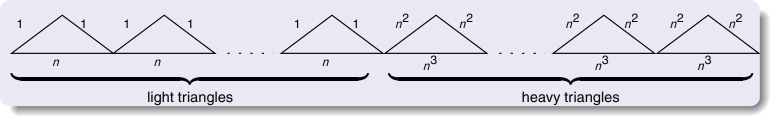
Solution (Wegener, 2005): MSTs are such an example.

A bad instance for MA



Simulated Annealing Beats Metropolis in Combinatorial Optimisation
Results



Theorem (Wegener, 2005)

The MA with arbitrary temperature computes the MST for this instance only with probability $e^{-\Omega(n)}$ in polynomial time. SA with temperature $T_t := n^3(1-\Theta(1/n))^t$ computes the MST in $O(n \log n)$ steps with probability $1 - O(1/\mathrm{poly}(n))$.

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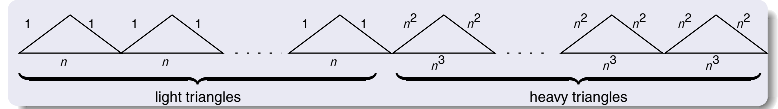

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Proof idea: need different temperatures to optimise all triangles.

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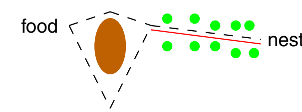
Ant Colony Optimisation — A Modern Search Heuristic
Background and Motivation

Ant colonies in nature



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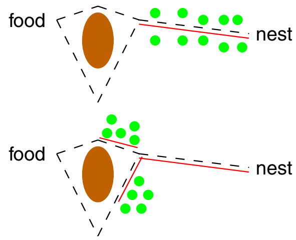

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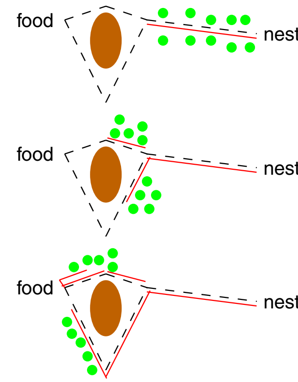

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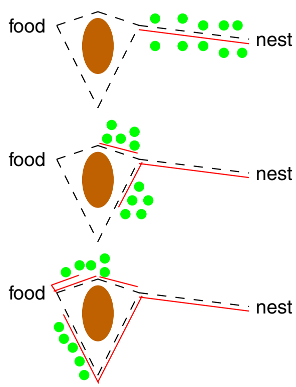


Ant Colony Optimisation (ACO) is yet another biologically inspired search heuristic.

Applications: combinatorial optimisation problems, e. g., TSP

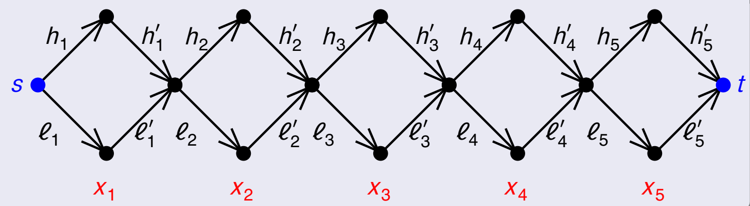
1-ANT for Pseudo-Boolean Optimisation

1-ANT

- Simple ACO algorithm
- Previously studied w. r. t. convergence
- Find maximum for pseudo-Boolean function $f\colon \{0,1\}^n \to \mathbb{R}$

1-ANT for Pseudo-Boolean Optimisation

Construction graph ($n = 5$)



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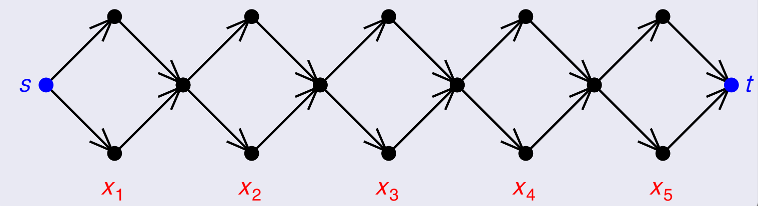

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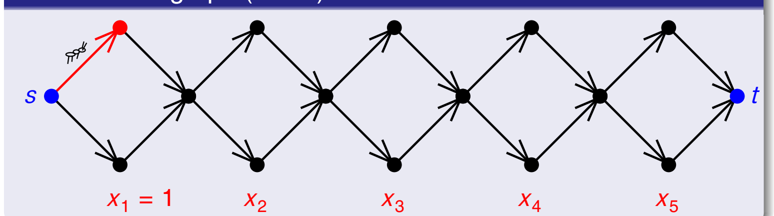

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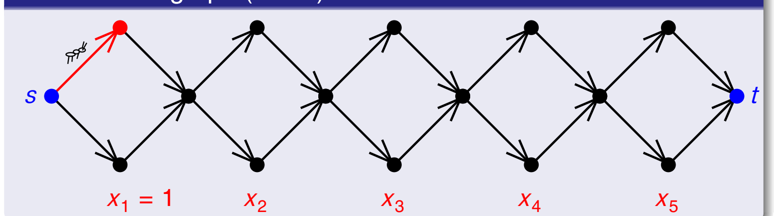

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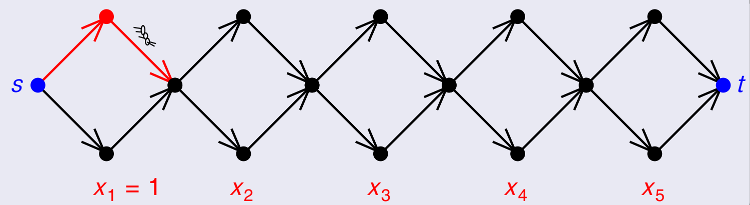

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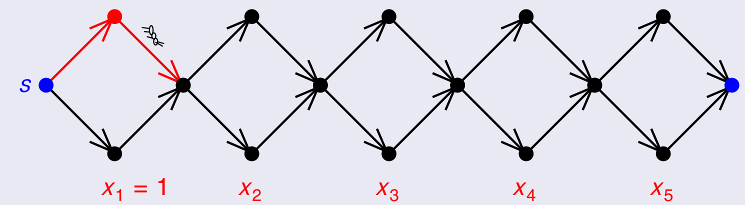

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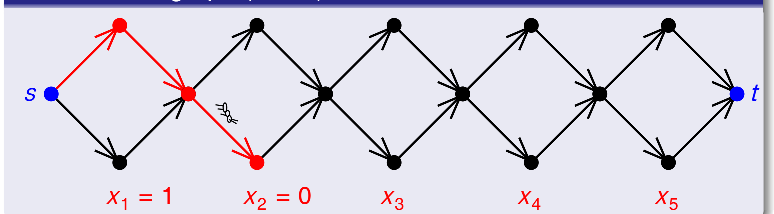

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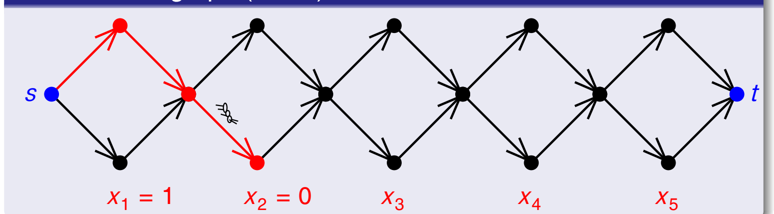

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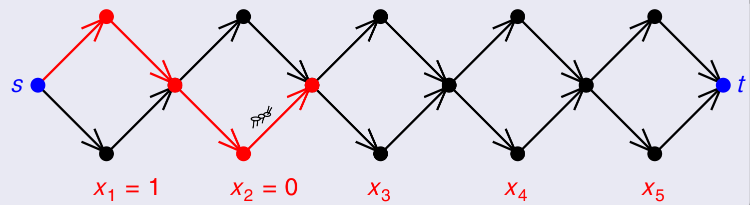

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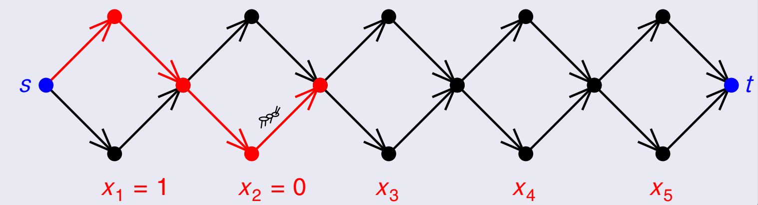

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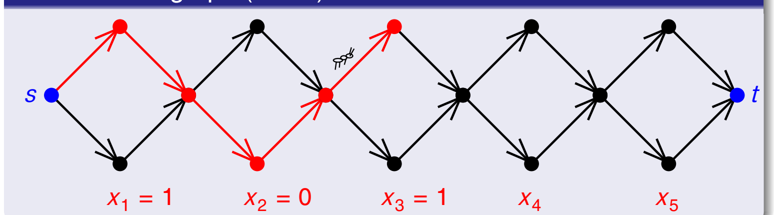

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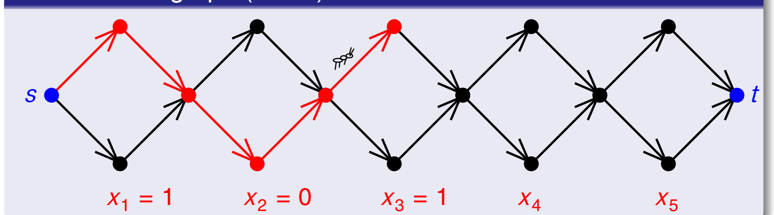

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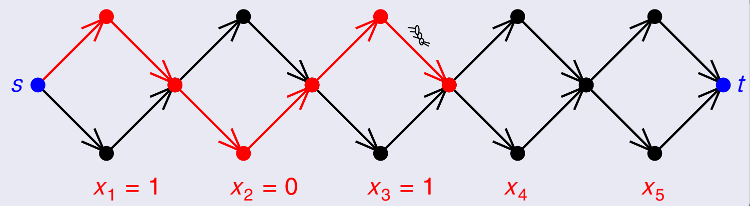

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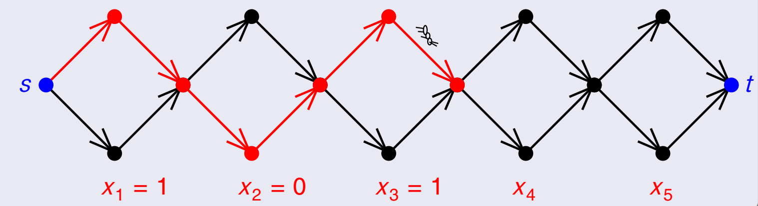

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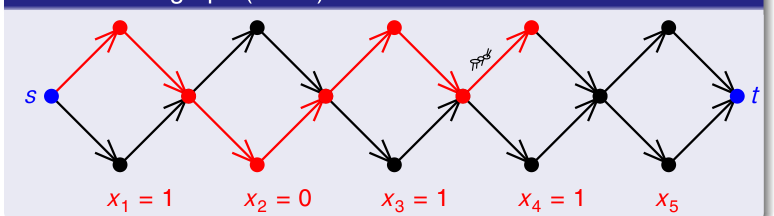

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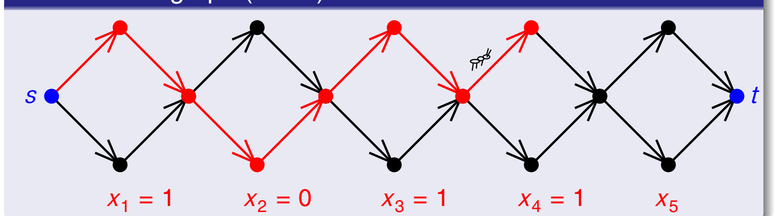

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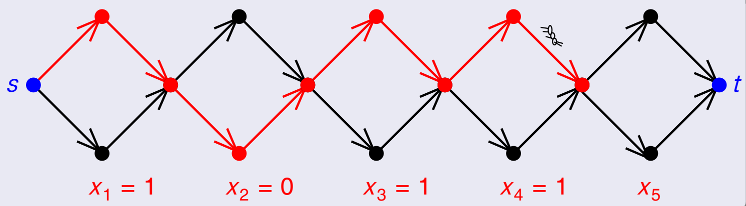

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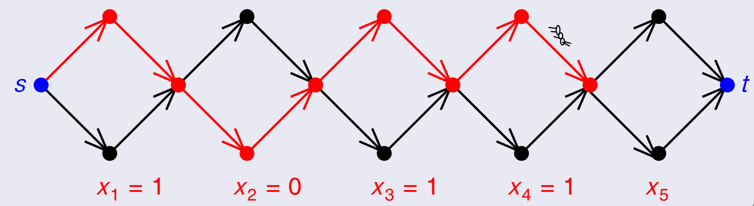

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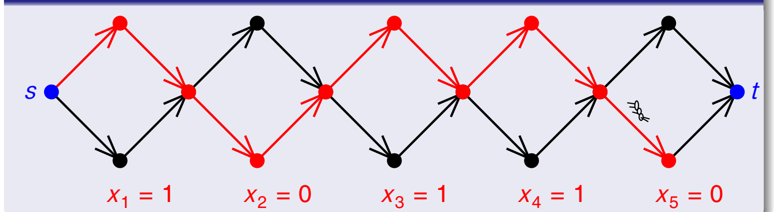

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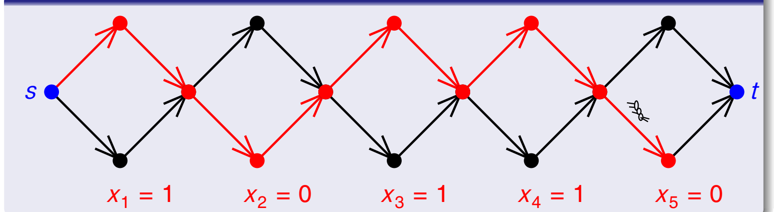

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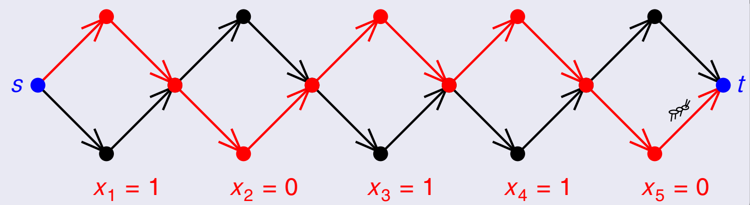

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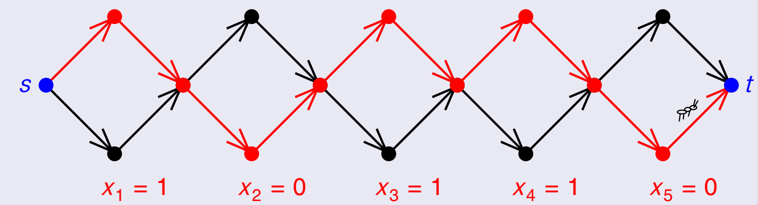

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1-ANT – Outline

Conventions

- Pheromone values = probabilities
- Upper and lower bounds for pheromone values
- Runtime = # constructed solutions until optimum found

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Algorithm 1-ANT for functions $f\colon \{0, 1\}^n \to \mathbb{R}$

- Set $\tau(e) = \frac{1}{2}$ for all edges e.
- Construct x (and $P(x)$), update pheromone; set $x^* := x$.
- Repeat
  - Construct x (and $P(x)$).
  - If $f(x) \geq f(x^*)$, update pheromone and set $x^* := x$.

1-ANT – Pheromone Update

- Crucial parameter: evaporation factor ρ, $0 \le \rho \le 1$
- Edge e is updated according to

$$e \in P(x) \quad \Rightarrow \quad \tau(e) := \min\left\{(1-\rho)\cdot\tau(e)+\rho,\, 1-\frac{1}{n}\right\}.$$
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- $\tau(h_i) + \tau(\ell_i) = 1$ for $1 \le i \le n$, i. e., probabilities
- Upper and lower bounds ensure that all probabilities in $[1/n, 1-1/n]$.

1-ANT: Runtime Analyses

- Simple but crucial: 1-ANT generalises (1+1) EA (just choose ρ large enough to keep all pheromone values in $\{1/n, 1-1/n\}$).

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- Results depending on ρ:

	superpol. runtime	poly. runtime
OneMax	$\rho = o(1/\log n)$	$\rho = 1 - O(n^{-\varepsilon})$
LeadingOnes	$\rho \le c_1/\log n$	$\rho \ge c_2/\log n$
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(Neumann/Witt, 2006; Doerr/Neumann/Sudholt/Witt, 2007; Doerr/Johannsen, 2007)

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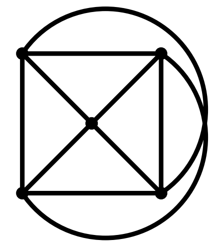
- Phase transitions: 1-ANT is not robust w. r. t. ρ
- Interesting for proofs: need inverse of concentration inequalities (old result by Hoeffding)

1-ANT for the MST problem
1st Construction Graph

- Algorithm by Broder (1989): uniformly generate spanning trees by random walks on graphs
- Random walk uniformly chooses a neighbour. If unvisited, add edge to spanning tree
- Algorithm stops after expected $O(n^3)$ steps (cover time).



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→ next constructed tree similar, but also likely to be better

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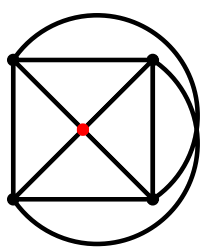

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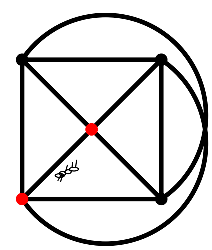

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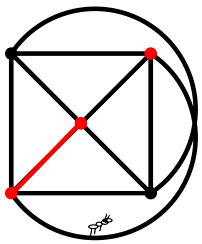

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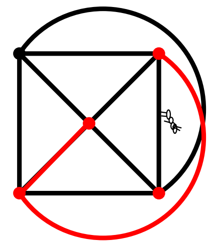

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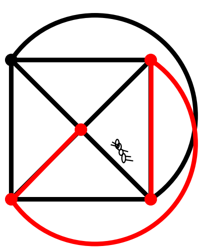

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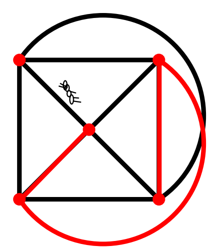

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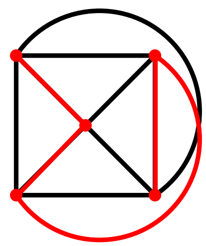

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2nd Construction Graph

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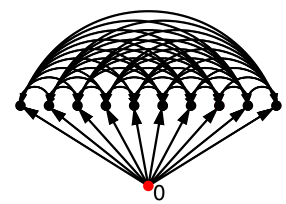
- Here: components = edges $\longrightarrow$ canonical construction graph $C(G) = (N, A)$ with $N = \{0, \dots, m\}$ (start node 0) and $A = \{(i, j) \mid 0 \le i \le m, 1 \le j \le m, i \ne j\}$.



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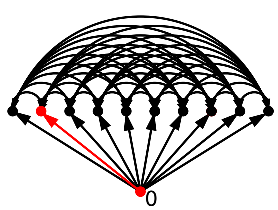

For path $v_1, \dots, v_k$ allowed neighbourhood $N(v_1, \dots, v_k) :=$ $(E \setminus \{v_1, \dots, v_k\}) \setminus \{e \in E \mid$ $(V, \{v_1, \dots, v_k, e\})$ contains cycle$\}$ (problem-specific aspect of ACO).

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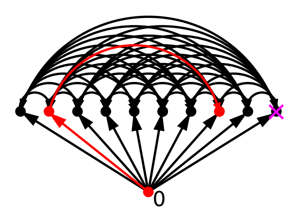

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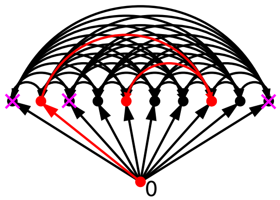

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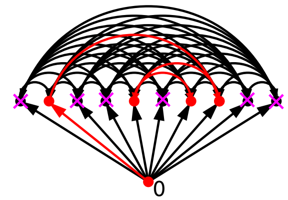

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1-ANT for the MST Problem
Results

Theorem (Neumann/Witt, 2006)

The expected number of constructed solutions until the 1-ANT with the 1st construction graph finds an MST is $O(n^6(\log n + \log w_{\max}))$

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Theorem (Neumann/Witt, 2006)

The expected number of constructed solutions until the 1-ANT with the 2nd construction graph finds an MST is $O(mn(\log n + \log w_{\max}))$.

Better than the (1+1) EA!

Summary and Conclusions

- Analysis of RSHs in combinatorial optimisation
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→ The analysis of RSHs is an exciting research direction.

Thank you!

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