The Harmonic Decision Matrix: a group of operators for the fuzzy-logic, multi-objective decisions and optimizations

Benoit Bagot

ZF Getriebe GmbH, Entwicklung Steuerungen PKW AT, Applikation Automatikgetriebe
PF 2567, 88015 Friedrichshafen, Germany
Benoit.Bagot@zf.com

Abstract. At the crossroad between fuzzy-logic, decision science, games theory, design of experiment, evolutionary algorithms and Pareto ranking, a new formula and its graphic representation helps to solve multi-objective problems; with possible applications in the development of complex technical systems, economics and many other situations. Used in a genetic program for the optimization of an automatic gear box, this tool helps to conciliate numerous, partly opposing criteria, in order to emphasize a unique final solution.

Keywords. utility function, real-world application, fuzzy systems, decision making, multi-objective optimization.

1 Introduction

A cat is either alive or dead, says the classic logic. When Schrödinger, in an imaginary experiment, considers a cat that is alive and dead, it is no longer classic mechanics, but it still deals with classic logic. Fuzzy-logic shows, however, how a smooth continuity can link these two operators, AND and OR. Whereas physicians try to understand the borderline phenomenon of decoherence, we studied the changes that occur by moving from an operator to another. In this paper, we will propose both a mathematical and a graphic tool with which this continuity can be better understood and used.

This tool was developed while working with an evolutionary algorithm (EA) from Pohlheim [13] and used for this purpose. Zadeh's observation [16] is therefore confirmed: "A trend that is growing in visibility relates to the use of fuzzy logic in combination with neurocomputing and genetic algorithms". We counted at least five good reasons for using decision science in EAs:

- Fuzzy rules, based on specific knowledge, can perform optimizations and can be combined to EAs in hybrid algorithms. Whereas optimization with fuzzy rules converges very quickly but needs prior information, EAs need less preparation and are more capable to explore original solutions but necessitate many optimization runs with a great number of individuals.
- Indeed, the multi-objective evolutions strategies in EAs allow working in a development phase without having to compromise. However, at the end of an

optimization run, a final, generally unique decision becomes necessary: not all the solutions on the last Pareto-front can be retained for serial realization. The decision maker may also interact with the EA between iterations (one or several generations) by "zooming in on the region of the Pareto set of interest", see Fonseca and Fleming [5].

- On the other hand, by an optimization with goals, a hard limit excludes compromises and may be abandoned in favor of a smooth constraint, where exceptions exist. In similar terms, Tan et al. [15] distinguished soft and hard priorities.
- Furthermore, the unification of several criteria in a global assessment represents a good method to simplify multi-objective optimizations in single- or few-objective problems; it becomes particularly necessary by a great number of objectives (e.g. 12 criteria in a population of 100 individuals), where Pareto ranking fails because nearly all individuals (98% in our example if we take random values) are Pareto-optimal. In real world applications, simple scalarization methods like the weighted-sum or the target vector do not reflect the subtlety of human decisions. And former publications proved that a unique assessment is even compatible with a need for diversification, if combined with techniques like clustering (Morse [11]) or fitness sharing (Goldberg and Richardson [7] and Zitzler [17]).
- In addition to this not exhaustive list, we mention a fifth important reason in favor of fuzzy-logic in EAs given by Jaszkiewicz [9]: "real-life multiple objective combinatorial optimization problems often include many parameters whose values are not known precisely".

Many books, papers, dissertations, and internet sites describe the principle of fuzzy logic, which begins with the selection of membership functions. We will limit ourselves here to a very short, incomplete reiteration in sections 2, 3.1 and 3.3. To compare with previous fuzzy operations, prior knowledge of fuzzy logic is assumed in this paper but is not absolutely required to understand the new tool. In section 3.2 we will present the new operator in its mathematic expression, in section 3.4 as a graphic picture. In section 4, the tool (a utility function) resolves an economic problem known as Allais' paradox, which we have chosen to illustrate the possible applications. Section 5 compares graphically the new ranking method with other possibilities and section 6 enlarges the decision procedure with the commonly used decision trees.

2 Membership and Assessment

The fuzzy logic as developed by L. Zadeh begins with the so-called fuzzification: a variable or quantity x is assigned a membership function $\mu(x)$. The variable is therefore normalized within a limited range. Usually the limit [0 1] is taken, we have chosen [0 10] in this paper.

The membership function can be discretely represented in the form of a pair of values or analytically as a mathematical function, like in formulas (1), corresponding to the first example in equation 1.

$$\text{patternfactor} = (\log(1/19)-\log(19)) / (x_{min} - x_{max}); \quad x_{mean} = (x_{min} + x_{max}) / 2; \tag{1}$$

$$\mu(x) = 10 / (1+\exp(\text{patternfactor} *(x_{mean} - x)));$$

Different kind of often used membership functions are represented in Fig. 1. When a general pattern has been chosen, the characteristic values (parameters) can be determined, to position the curve on the x scale.

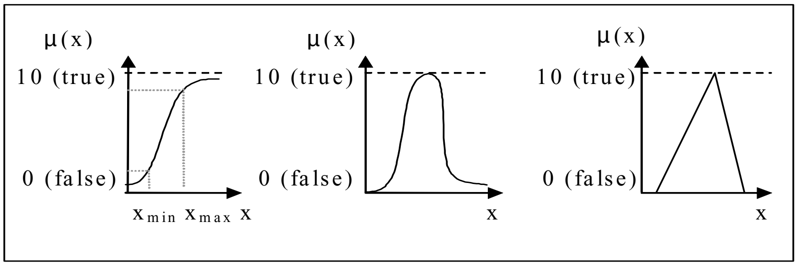

Fig. 1. Membership functions: examples

We can not only define one membership scale (e.g. warm/not warm) for each ascertainable variable, but also one or more assessment scales selected for these variables. Think of the decision theory:
True/false, relevant/irrelevant, interesting/uninteresting, good/bad, correct/false, advantage/disadvantage, red/black numbers, less risky/very risky, certain/uncertain, soon/later, current/obsolete, secure/insecure information, fear/hope, life/death, luck/misfortune, sad/happy, boring/exciting, visionary/empirical, etc.

The first couple (true/false) represents, in this generalized logic with a strong subjective character, just one out of numerous possibilities, without particular status. This way, from a base variable like a physically measured value or also a subjective classification, we can derive one or more normalized values. For example, a temperature value can be defined as warm on a first scale, as uncertain on a separate scale, and as little interesting on a third one.

3 Operators

The quantities normalized with the help of membership functions are linked together by operators (rules from a knowledge-based science) in the second stage of the fuzzy method. Then the various rules are linked together themselves (composition) and "defuzzyfied" in a third stage. In this paper we are especially concerned with the linking operators.

3.1 Several known Operators

We listed several operators in a table (2), which were already collected with additional operators from Richter [14]. It is difficult to imagine what they really mean if you are seeing them for the first time. It is even hardly adequate at first to point out that the formulas denoted as "products" are related to an AND, while the "sum" resembles an OR. Therefore, in the following section, we suggest not only a new formula, but also an associated graphic image.

AND

$$\mu(\mu_1(x), \mu_2(x)) = MIN(\mu_1(x), \mu_2(x)) \quad (2)$$

OR

$$\mu(\mu_1(x), \mu_2(x)) = MAX(\mu_1(x), \mu_2(x))$$

Hamacher Product

$$\mu(\mu_1(x), \mu_2(x)) = \mu_1(x)*\mu_2(x) / (\mu_1(x)+\mu_2(x)-\mu_1(x)*\mu_2(x))$$

Hamacher Sum

$$\mu(\mu_1(x), \mu_2(x)) = (\mu_1(x)+\mu_2(x)-2*\mu_1(x)*\mu_2(x)) / (1-\mu_1(x)*\mu_2(x))$$

Einstein Product

$$\mu(\mu_1(x), \mu_2(x)) = \mu_1(x)*\mu_2(x) / (1+(1-\mu_1(x))*(1-\mu_2(x)))$$

Einstein Sum

$$\mu(\mu_1(x), \mu_2(x)) = (\mu_1(x)+\mu_2(x)) / (1+\mu_1(x)*\mu_2(x))$$

3.2 The Harmonic Decision Matrix

In order to simplify the notation here, we use $\mu_1(x) = X$, $\mu_2(y) = Y$, $\mu(\mu_1(x), \mu_2(y)) = Z$; f_1 and f_2 are monotonic functions. As results of the membership functions we have $X \in [0\ 10]$; $Y \in [0\ 10]$; Z is defined as follows in equation (3)

$$\forall(X,Y),\ (X+f_1(X)-Z)(Y+f_2(Y)-Z)-f_1(X)*f_2(Y)=0 \quad (3)$$

$$\text{with } Z \in [0\ 10],\ Z(X,X)=X,\ Z \text{ monotonic}$$

The definition may also be expanded to higher dimensions in an equation (4):

$$\prod_{n=1}^{N}(X_n+f_n(X_n)-Z) - \prod f_n(X_n)=0 \quad (4)$$

This group of functions belongs to a general family F(X,Y), which fulfills the following conditions: $(X,Y) \in [0,0]*[0,0]$, $F(X,Y) \in [0,0]$; $F(X,X)=X$; F is monotonic.

The condition F(X,X)=X means simply that sub-assessment 7 and sub-assessment 7 result in an overall assessment 7, that sub-score 5 and sub-score 5 result in an overall score 5, etc. As depicted, it seems straightforward. Actually it is, theoretically, not absolutely necessary, but very easy in practice, to show sub and total scores on the same scale. With this selection it becomes clearer which sub-scores contribute to an improvement in the total score and which to deterioration. A combination X,X (two identical sub-assessments) is denoted as balanced or harmonious. At a combination of Y>X, one will emphasize, depending on attitude and mood, Y (positive attitude) or X (negative attitude), meaning praise or criticism.

The condition F monotonic is equally contestable but it applies in many situations. It means that the total assessment scores higher as soon as a sub-assessment scores higher, everything else remaining equal. However, in some cases the following takes place: An object, which aside from a property assessed as average also possesses another very good feature, seems unbalanced and is criticized more often than the comparable object with only average qualities. In order to take this possibility into consideration, the equilibrium (score deviation between sub-assessments) can simply be used as an additional criterion. Therefore the condition F remains monotonic.

When one wants to optimize a total solution using the fuzzy AND, it is much more effective to work closely to the best fit straight line (Y=X). This means that it would be best if all sub-criteria were drawn up at the same time. Using fuzzy OR one can also effectively work if one criterion is focused on. A natural but imprecise statement: It can be quantified with the harmonic decision matrix.

Example "fuzzy AND":

$$f_1(X) = k_x * X; \text{ with } k_x = \text{constant}$$
$$f_2(Y) = k_y * Y; \text{ with } k_y = \text{constant} \quad \textbf{(5)}$$
$$Z = (X+f_1(X)+Y+f_2(Y))/2$$
$$-\text{sqrt}((X+f_1(X)+Y+f_2(Y))^2 -4*((X+f_1(X))*(Y+f_2(Y))-f_1(X)*f_2(Y)))/2;$$

Example "fuzzy OR":

$$f_1(X) = -k_x * (10-X); \text{ with } k_x = \text{constant}$$
$$f_2(Y) = -k_y * (10-Y); \text{ with } k_y = \text{constant} \quad \textbf{(6)}$$
$$Z = (X+f_1(X)+Y+f_2(Y))/2$$
$$+\text{sqrt}((X+f_1(X)+Y+f_2(Y))^2-4*((X+f_1(X))*(Y+f_2(Y))-f_1(X)*f_2(Y)))/2;$$

3.3 Result-Oriented and Variable-Oriented Representations

A variety of possible representations were already suggested in the context of fuzzy logic, see fig. [2] and [3] on the right. Inspired by the work with the Pareto ranking, we used isocurves or isosurfaces, which divide a two- or multi-dimensional space into several zones according to the rank value, as shown in Fig [3] on the left.

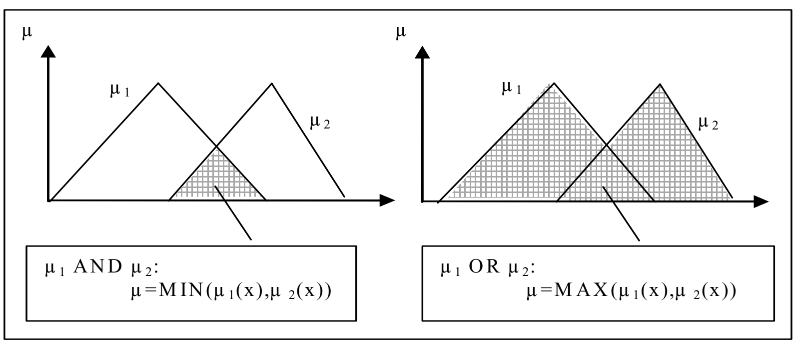

Fig. 2. Result oriented representation

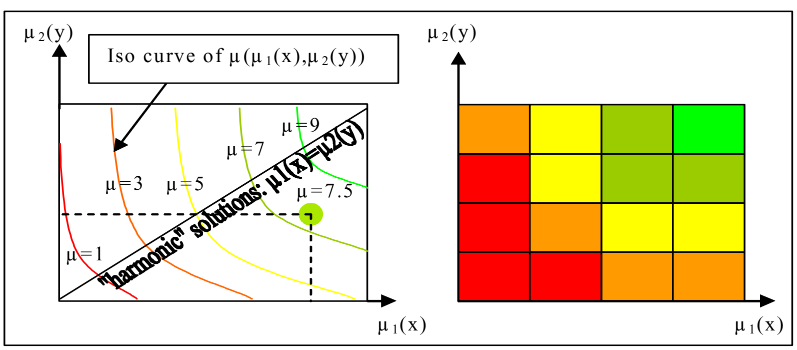

Fig. 3. Variable oriented representation

3.4 AND, OR, AT LEAST Graphic with the Harmonic Decision Matrix

The condition F(X,X)=X does not mean that F must be symmetrical, meaning ∀ [X,Y] F(X,Y)=F(Y,X). Exaggerated, the asymmetry leads to a new operator, which we call AT LEAST. Colloquially, one can express it as follows: Above all, I want X, if I also receive Y, that's even better.

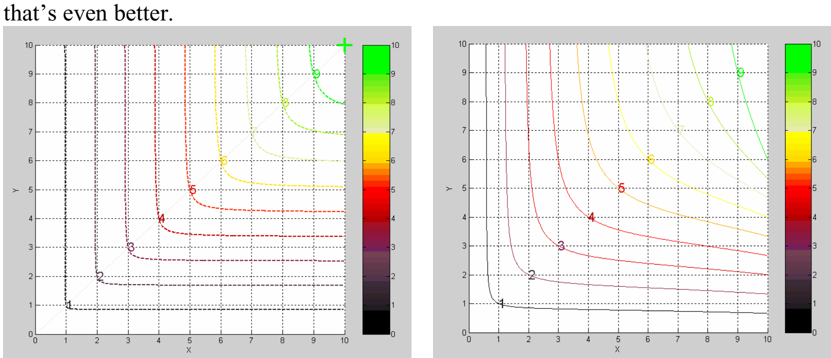

Fig. 4. Fuzzy AND

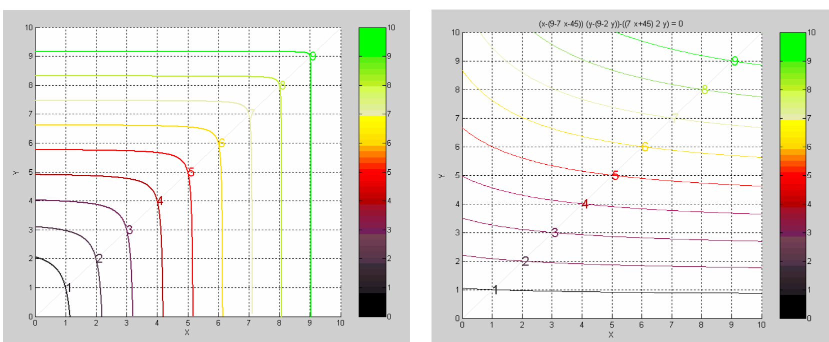

Fig. 5. Fuzzy OR (left), fuzzy AT LEAST (right)

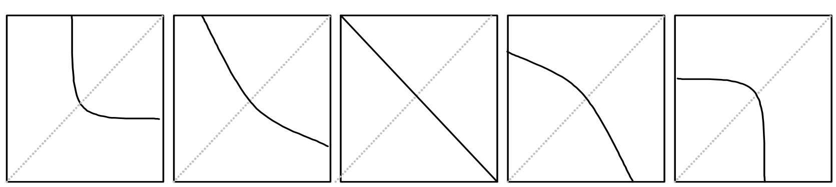

Fig. 6. From AND to OR in small steps. In the middle: the mean value

In section 3.2 the equation (4) expands the mathematic principle to higher dimensions without apparent difficulty. For the graphic representations however the third dimension

(Fig. 7) already requires much more precaution. Therefore, if more than 3 dimensions join together, it is recommended to structure them (section 6), in order to treat them in small groups. Or the problem is simplified with the use of minimum, maximum, mean or weighted values that are easier to manipulate.

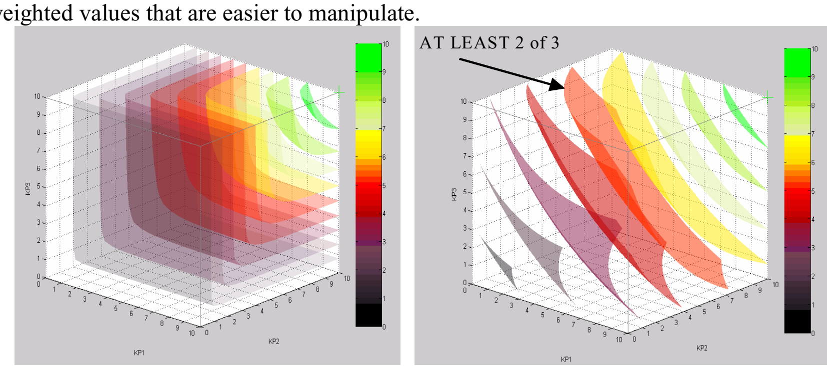

Fig. 7. Fuzzy AND in 3D (left), AT LEAST 2 of 3 (right)

4 Refutation of the Independence Axiom of the Bernoulli Principle

Originally we designed this special decision matrix in the context of the development of an automatic transmission, therefore a technical product. We wanted to optimize several attributes, such as comfort and spontaneity, with an evolutionary process. Because we wanted to select only one solution at the end of the optimization, an overall assessment was absolutely required.
We then found additional application examples not only in technical systems, but also in business management problems (economics) and finance. It was an exciting experience to realize that this tool can actually support decisions in many of life's situations.

For example, the paradox of Maurice Allais, which R. Meyer [10] described very well, can now be deciphered. In this decision paradox, one single alternative must be chosen two times from several alternatives. It is therefore a ranking problem where the alternatives A and B, as well as C and D (Table 1), must be sorted according to personal values.

Table 1. Alternatives

Alternative A	One receives 100 million with certainty (A)
Alternative B	One has 10 chances out of 100 to win 500 million (B1)
	One has 89 chances out of 100 to win 100 million (B2)
	One has 1 chance out of 100 to win nothing (B3)

xxx

Alternative C	One has 11 chances out of 100 to win 100 million (C1)
	One has 89 chances out of 100 to win nothing (C2)
Alternative D	One has 10 chances out of 100 to win 500 million (D1)
	One has 90 chances out of 100 to win nothing (D2)

According to the independence axiom of the Bernoulli principle, the problem can be presented in tabular form as follows in table 2:

Table 2. Alternatives according to the independence axiom

	A	B	C	D
89	100	100	0	0
10	100	500	100	500
1	100	0	100	0

This should imply that if A>B, then C>D. Today, empirical tests demonstrate that as a rule A is indeed preferred to B, but D is preferred to C. Everyone can test his own choice. This paradox is convincingly illuminated with the harmonic decision matrix.

The objective is clear: a solution with big winnings AND big chances. One takes:

X = Chances/10 = [10 1 8.9 0.1 1.1 8.9 1.0 9.0];

y = Winnings = [100 500 100 0 100 0 500 0]

Y = normalized gain =10/(1+exp(0.0076*(55843/y-y)))=[3 9.5 3 0 3 0 9.5 0];

As AND Function: μ(X,Y) = sqrt(X*Y) corresponding to X*Y in [0,1]*[0,1]

Thus the following overall values result for the individual possibilities:

[A	B1	B2	B3	C1	C2	D1	D2]
[5.48	3.08	5.17	0	1.82	0	3.08	0]

A = 5.48 > (B1+B2+B3)/3 = 2.75;

(D1+D2)/2 = 1.54 > [C1+C2)/2 = 0.9

The paradox also does not apply in the graphic presentation Fig. 8. In fact, we took a well known operator in this case, our work just helped to visualize the problem.

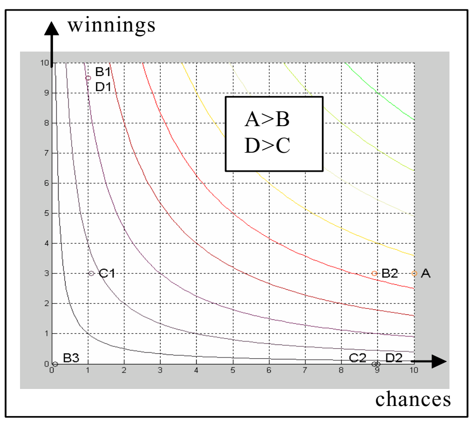

Fig. 8. Allais Paradox in a fuzzy AND graphic

The Petersburg Paradox [3] can be processed in a similar way. For this problem the normalized time (when can I cash in my winnings: Immediately=10, infinity=0) on one axis and the normalized winning levels on the second axis. The objective in this case is: Big winnings AND immediate pay out.

5 Strategies. Absolute assessment and population-related ranking.

In this section we want to consider the harmonic decision in a larger context. The graphic representation helps us to compare it with many other strategies. For the last two decades, evolutionary algorithms have been considerably growing in complexity. Benchmark EAs and multidisciplinary optimization make profit of many different strategies that exist next to each other, like different size balls in a glass that we continue to fill with our small contribution. It is impossible to report all the examples given from recent literature in this paper. As diversified as strategies, there are many ways to represent the same object. In this section we chose a 2D graph to represent what Fonseca and Fleming [6] called the "cost landscape". The axes represent either objective values, term used here as target rather than in opposition to subjective, or parameters of the system to optimize that can influence these objectives; the points show single individuals of a population. We want to demonstrate some known strategies, based on the phenotype (all the objective values) or on the genotype (all the parameters) and corresponding ranking methods. In the following examples, we also brought together two different types of ranking: absolute ranking, only depending on measurable variables and population related or fitness ranking, also depending on the real distribution in a population.

- The goal-oriented strategy: a goal is striven for, taking into account several aspects, a harmonized decision matrix type "fuzzy UND" or "fuzzy AT LEAST" is used. This strategy represents the heroes, who can do everything good and have advantageous resources at their disposal in every situation.
- The multi-criteria strategy: multiple objectives are pursued without signifying a preference. In this case, a "fuzzy OR" or a Pareto ranking process is used. The strategy of the pragmatist and specialist: obtain what one can; get the best in one field.
- The expansion strategy and the unity strategy: under the expansion strategy, one searches for solutions which are as far away as possible from the existing examples with the hope that something positive comes out of it. This is the strategy of creativity, of discovery, of research and development, of diversity. Strikingly good, strikingly bad, most importantly striking, is the motto of the extravagant person. Under the cohesion strategy or unity strategy, one inversely tries to find a consensus, which reduces the dispersion. Strengthening the group, simplification, predictability, quality in a mass production is preferred here.
- The systematic exploration, niche strategy and clustering: at the full-factorial and, for example, at the D-optimal or Taguchi strategy, the space of all possibilities is covered with one network, also called Design of Experiment. Full-factorial and D-optimal are generally not related to the criteria (phenotype) of an existing population but used to create the parameters (genotype) of a first population at the beginning of an evolutionary algorithm, replacing a random function. Niches are defined here as the small holes that one can fill in one's immediate surroundings. Clustering can be associated to this group as a kind of network, where the holes rather than the knots are considered.
- Additional strategies: the following are supplementary but infinitely incomplete.

– the strategies of the traveling salesman,
– the strategy of the city wall, whose external points are reinforced,
– The jittering (U. Fayyad, [4]), stochastic movements of small amplitude.
– The separation, the fusion, the repetition

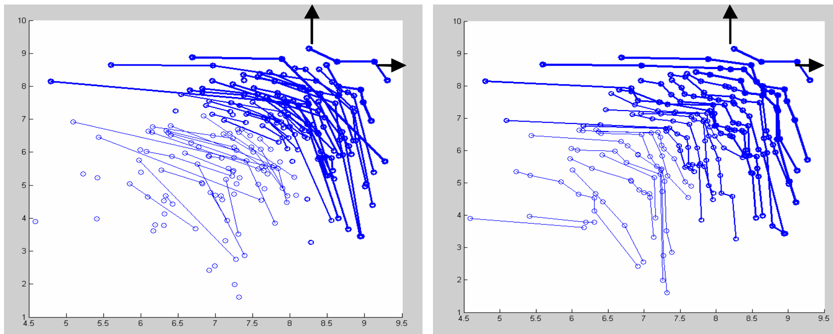

Fig. 9. Pareto ranking, based on phenotype. Fonseca [5] (left) or Goldberg [8] (right) Compare with “fuzzy OR” fig [5]

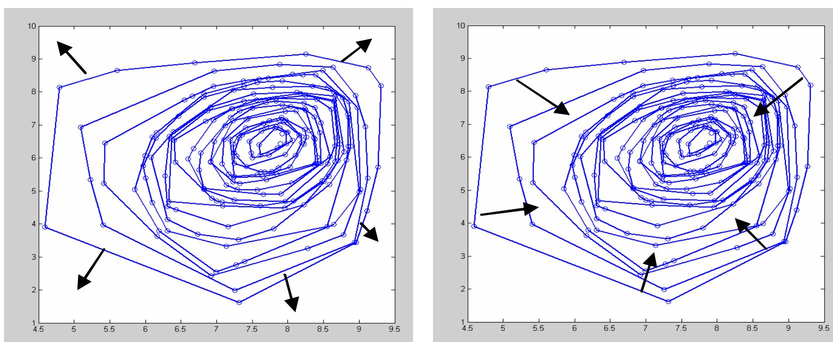

Fig. 10. Convex hull. Expansion (left), cohesion (right)

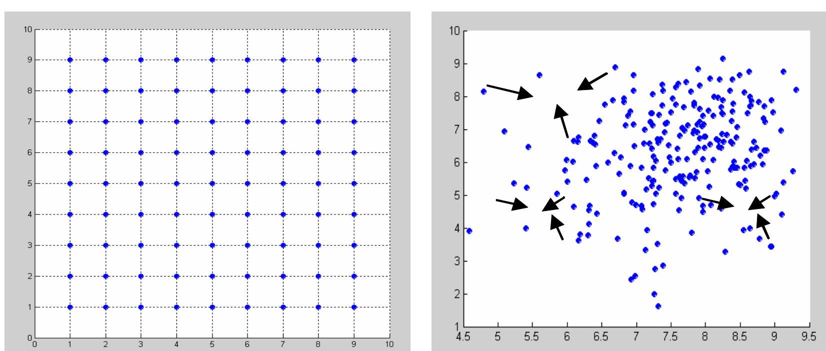

Fig. 11. Design of Experiment (left), niche (right). For other details and definitions concerning niching techniques (fitness sharing, crowding), see Zitzler and Thiele [17]

Of course, several strategies can be used in combination like in Fig.12.

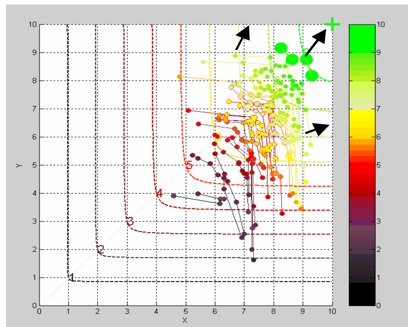

Fig. 12. Combination fuzzy AND and Pareto ranking

6 Procedures

If such methods and strategies are used in a program, a systematic procedure is recommended. For complex decision problems, one proceeds as follows:

- Separate the problem in all important sub-problems and create a tree, so that the overall decision will result from several sub-decisions.
- Normalize all base variables of the decision problem in order to better compare "apples and oranges."
- Choose the appropriate logic at each level of the tree.
- Apply the chosen rules for each alternative: One single overall score will result for each alternative. This stage can be automated.
- Compare overall scores of all alternatives.

If we train with the general assessment of a document (this one) we take this method to create a decision tree. Of course, the criteria can be completed towards the right.

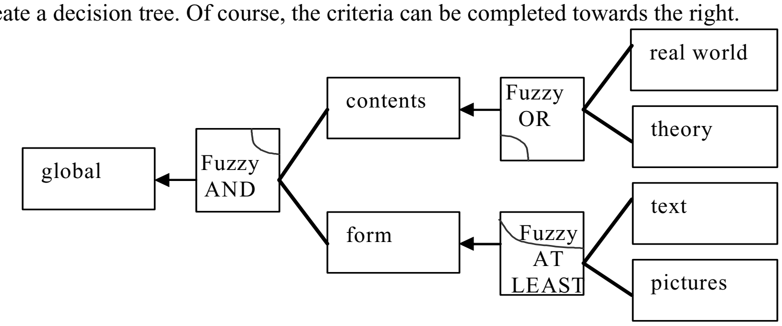

Fig. 13. Decision tree for document assessment

There will be additional examples and details in the software technology textbooks. Balzert [1] dedicates a long chapter to this. We stress once again the advantage of normalizing several completely different criteria on one collective scale. It presupposes that one has clearly organized one's subjective, personal values.

7 Conclusion

You can judge for yourself whether individual decisions come easier with the harmonized decision matrix. “The computer answers our questions. It does not ask them. […] do we refrain from rejecting numbers if they allow more exactness than those marked with incoherent intuition or reflexes” (P. Bernstein, translated from [2]); mathematic equations and clever diagrams can definitely help to describe decisions in detail. This is important in order to find more acceptance in group decisions.

The discussion whether people decide rationally or irrationally has elicited many interesting results, but did not result in any final answer. This remains true today. A big advantage of objectifying decision lies in the possibility of using strategies systematically in a repeatable and even automated process. The relief that results from this can free up more capacities to search for new strategies.

References

[1] *Balzert, H*. (2000): Lehrbuch der Softwaretechnik. Software-Entwicklung. 2. Auflage, Spektrum Akademischer Verlag, p. 270: Entscheidungstabellen und Entscheidungbäume.

[2] *Bernstein, P. (1998).*: Plus forts que les dieux. La remarquable histoire du risque (Original: Against the gods: the remakable Story of Risk) Flammarion, p. 319.

[3] *Bernstein, P. (1992)*: Des idées capitales, (Original: Capital Ideas) Presses Universitaire de France, p. 157: the Petersburg Paradox (Daniel Bernoulli).

[4] Fayyad*, U. (2002)*: Visualization in Data Mining and Knowledge Discovery, Academic Press, p. 49: Jittering

[5] Fonseca*, C.M. and Fleming, P.J. (1993)*: Genetic Algorithms for Multiobjective Optimization: Formulation, Discussion and Generalization.

[6] Fonseca*, C.M. and Fleming, P.J. (1994)*: An Overview of Evolutionary Algorithms in Multiobjective Optimization.

[7] *Goldberg, D.E. and Richardson, J. (1987)*: Genetic algorithms with sharing for multimodal function optimization.

[8] *Goldberg, D.E. (1989)*: Genetic Algorithms in Search, Optimization, and Machine Learning.

[9] *Jaszkiewicz, A. (2001)*: Multiple Objective Metaheuristic Algorithms for Combinatorial Optimization, p.69.

[10] *Meyer, R. (1999)*: Entscheidungstheorie, Gabler Verlag, p. 66: Empirische Widerlegung des Unabhängigkeitsaxioms durch ALLAIS.

[11] *Morse, J.N. (1980)*: Reducing the size of the nondominated set: Pruning by clustering. Computers and Operation Research.

[12] *Pohlheim, H. (1999)*: Evolutionäre Algorithmen - Verfahren, Operatoren, Hinweise aus der Praxis. Berlin, Heidelberg: Springer-Verlag. http://www.pohlheim.com/eavoh/, p. 21 (Mehrkriterielle Fitneßzuweisung), p. 79 (Topologie von Nachbarschaften)

[13] *Pohlheim, H. (*1994-2004): *GEATbx* - Genetic and Evolutionary Algorithm Toolbox for *Matlab*. http://www.geatbx.com/

[14] Richter *, R. (1994)*: Planung und Steuerung von Entscheidungsprozessen mit Fuzzy-Petri-Netzen, http://nestroy.wi-inf.uni-essen.de/~conen/conen/da/richter/fpn3f.doc, p. 16: Operatoren auf unscharfen Mengen.

[15] *Tan, K. C., Khor, E. F., Lee, T. H. and Sathikannan, R. (2003)*: An Evolutionary Algorithm with Advanced Goal and Priority Specification for Multi-objective Optimization.

[16] *Zadeh, L. (2001)*: Mathworks User’s Guide Version 2. Fuzzy Logic Toolbox for Use with Matlab, Foreword.

[17] *Zitzler, E. and Thiele, L. (1998)*: An Evolutionary Algorithm for Multiobjective Optimization: The Strength Pareto Approach.