Walsh Transforms, Balanced Sum Theorems and Partition Coefficients over Multary Alphabets

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ABSTRACT

In this note, we indicate how the basic machinery of Walsh transforms can be generalized from the binary case ([3, 4]) to multary alphabets. Our main results show how Walsh coefficients are related to partition coefficients and how they may be used to calculate schema averages.

Categories and Subject Descriptors

I.2.8 [**Artificial Intelligence**]: Problem Solving, Control Methods, and Search; F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems.

General Terms

Algorithms, Theory.

Keywords

Genetic algorithm, schema theory, Walsh function, Walsh coefficient, partition coefficient, schema average, multary alphabet.

1. INTRODUCTION

It is well-known that the ordinary Fourier transform is extremely well-suited to study periodicity and density properties of real or complex valued functions. In the same way, its binary counterpart, the *Walsh transform* plays a fundamental role in the analysis of genetic algorithms (GAs)[5, 7,

11]. Indeed, they allow for a practical description of epistasis [13]. This notion of epistasis measures, roughly speaking, the presence of links between separate bits in the codified version of the data to be optimized. Although, epistasis alone in no way suffices to predict GA-hardness , it has been shown to correctly correlate to GA-hardness within several particular classes of fitness functions ([9, 13]).

Schemata (hyperplanes in the search space) automatically appear when fitness functions are constructed and the working of the crossover operator is considered. The *partition coefficients* $\varepsilon(H)$, associated to any schema H, have been used (see [1, 12]) in the analysis and construction of problems that are difficult to solve with a genetic algorithm and also to characterize static deception in a problem. Earlier papers by Goldberg [3, 4] and Heckendorn and Whitley [6] connected schemata and partition coefficients to the Walsh transforms revealing important properties for the understanding of GAs.

On the other hand, although binary encodings are "standard" in the context of GAs, sometimes it is much more natural to use a different type of encoding over not necessarily binary alphabets ([2, 10]). This work extends the results of [3, 4, 6] to functions over multary alphabets, i.e., functions over the space of strings $\Omega_n = \{0, \ldots, n-1\}^\ell$, with $n \geq 2$. Actually, our approach also yields more compact and elegant proofs for the aforementioned results.

2. GENERALIZED WALSH TRANSFORMS– FIRST APPROACH

We will denote by Ω_2 the set $\{0, 1\}^\ell$ of length ℓ binary strings - we fix ℓ throughout this text. We will sometimes identify elements $s = s_{\ell-1} \cdots s_0 \in \Omega_2$ with their numerical value, viewing s as the binary expression of some integer $0 \leq s < 2^\ell$.

To any $t \in \Omega_2$ one associates the so-called *Walsh function* ψ_t, defined by

$$\psi_t(s) = (-1)^{s \cdot t} = \prod_{i=0}^{\ell-1} (-1)^{s_i t_i},$$

where $s \cdot t = \sum_{i=0}^{\ell-1} s_i t_i$ is the scalar product of $s, t \in \Omega_2$. For example, $\psi_{11011}(01100) = -1$ and $\psi_{11011}(01010) = 1$. Note that $\psi_t(s)$ can be 1 or -1 depending on the number of loci where both s and t have value 1. Alternatively, define the conjunction $\wedge$ in Ω_2 by

$$s \wedge t = (s_{\ell-1} \wedge t_{\ell-1}) \ldots (s_0 \wedge t_0),$$

where $s_i \wedge t_i$ is just the product $s_i \cdot t_i$, for all $s_i, t_i \in \{0, 1\}$ and denote by $u(s)$ the number of loci with value 1 in any string s. Then it is easy to see that

$$\psi_t(s) = (-1)^{u(s \wedge t)}.$$

In order to generalize Walsh coefficients to a multary alphabet $\Sigma = \{0, 1, \cdots, n - 1\}$, we have at least two possibilities. The first one is inspired by the discrete Fourier transform. We consider the primitive root of unity $r = e^{\frac{2\pi}{n}i} = \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n})$ (note, that for $n = 2$, we have $r = -1$) and define, for any $s \in \Omega_n = \Sigma^\ell$, the Walsh function ψ_s on Ω_n by putting

$$\psi_s(t) = r^{s \cdot t},$$

for every $t \in \Omega_n$. Here $s \cdot t$ denotes the scalar product modulo n of s and t in Ω_n, i.e.,

$$s \cdot t = \sum_{i=0}^{\ell-1} s_i \cdot t_i \pmod{n}.$$

For example, if $n = 3$ and $s = 012$, $t = 121$, we have that:

$$\psi_s(t) = r^{(0+2+2) \pmod 3} = r,$$

where $r = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Obviously, for any $s, t \in \Omega_n$, it follows that

$$\psi_s(t) = \psi_t(s)$$

We may now prove the following multary version of the Balanced Sum Theorem (see [6] for the binary version).

THEOREM 1 (FIRST BALANCED SUM THEOREM). *For any* $t \in \Omega_n$, *we have*

$$\sum_{s \in \Omega_n} \psi_t(s) = \sum_{s=0}^{n^\ell - 1} \psi_t(s) = \begin{cases} n^\ell & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Clearly

$$\sum_{s \in \Omega_n} \psi_0(s) = \sum_{s \in \Omega_n} 1 = n^\ell$$

Now, if $t \neq 0$, suppose that $t_{\ell-1} \neq 0$, and denote, $s = s_{\ell-1}\widehat{s}$ and $t = t_{\ell-1}\widehat{t}$, with $\widehat{s}, \widehat{t} \in \Sigma^{\ell-1} = \widehat{\Omega}_n$. Then:

$$
\begin{aligned}
\sum_{s \in \Omega_n} \psi_t(s) &= \sum_{s \in \Omega_n} r^{s_{\ell-1}t_{\ell-1}} r^{\widehat{s}\,\widehat{t}} \\
&= \sum_{\substack{s \in \Omega_n \\ s_{\ell-1}=0}} \psi_{\widehat{t}}(\widehat{s}) + r^{t_{\ell-1}} \sum_{\substack{s \in \Omega_n \\ s_{\ell-1}=1}} \psi_{\widehat{t}}(\widehat{s}) + \cdots \\
&\quad + r^{(n-1)t_{\ell-1}} \sum_{\substack{s \in \Omega_n \\ s_{\ell-1}=n-1}} \psi_{\widehat{t}}(\widehat{s}) \\
&= \left(1 + r^{t_{\ell-1}} + \cdots + r^{(n-1)t_{\ell-1}}\right) \sum_{\widehat{s} \in \widehat{\Omega}_n} \psi_{\widehat{t}}(\widehat{s}) \\
&= 0.
\end{aligned}
$$

$\square$

If we denote by $\widetilde{s} = \widetilde{s}_{\ell-1} \ldots \widetilde{s}_0$, the string with $s_i + \widetilde{s}_i = 0 \pmod n$, for any $s = s_{\ell-1} \cdots s_0 \in \Omega_n$, it follows immediately from the previous result:

COROLLARY 1.

$$\sum_{s \in \Omega_n} \psi_t(s)\psi_{t'}(s) = \begin{cases} n^\ell & \text{if } t' = \widetilde{t} \\ 0 & \text{otherwise.} \end{cases}$$

Now, if we define, for any $t \in \Omega_n$ and any function $f : \Omega_n \to \mathbb{R}$, the corresponding *complex Walsh coefficient*

$$v_t^c = n^{-\ell} \sum_{s \in \Omega_n} f(s)\psi_t(\widetilde{s}) = n^{-\ell} \sum_{s \in \Omega_n} f(s)\psi_{\widetilde{s}}(t),$$

we can prove that the relation between the function f and these coefficients is given by:

$$f(s) = \sum_{t \in \Omega_n} \psi_t(s)v_t^c \tag{1}$$

for all $s \in \Omega_n$. Indeed, if $s \in \Omega_n$, then

$$
\begin{aligned}
n^\ell \sum_{t \in \Omega_n} \psi_t(s)v_t^c &= \sum_{t \in \Omega_n} \psi_t(s) \sum_{a \in \Omega_n} f(a)\psi_{\widetilde{a}}(t) \\
&= \sum_{a \in \Omega_n} \sum_{t \in \Omega_n} \left(\psi_s(t)\psi_{\widetilde{a}}(t)\right) f(a) \\
&= n^\ell f(s).
\end{aligned}
$$

3. GENERALIZED WALSH TRANFORMS - SECOND APPROACH

As the sum in the First Balanced Sum Theorem is taken over the whole space Ω_n, one may wonder what happens if one restricts s to a schema (or hyperspace) H in Ω_n. A schema is usually written as a string over $\{0, \ldots, n - 1, \#\}$. This is equivalent to H belonging to $\{0, \ldots, n - 1, \#\}^\ell$, where $\#$ plays the role of a wild-card symbol. An individual $s_{\ell-1} \ldots s_0$ belongs to a schema $H = h_{\ell-1} \ldots h_0$ if $s_i = h_i$ for all i such that $h_i \neq \#$. For example, if $n = 3$ and $H = 1\#20$, then the strings of H are:

$$H = \{1020, 1120, 1220\}$$

The order $o(H)$ of H is the number of fixed positions in H (the number of non-wild-card symbols in the schema). If we denote by $|H|$ the cardinality of H (the number of strings "in" or "satisfying" H), then $|H| = n^{\ell-o(H)}$. For the above schema $H = 1\#20$, we have $o(H) = 3$ and, as we have seen $|H| = 3^{4-3} = 3$.

In order to obtain another generalized Walsh transform, let us start by defining the partition coefficient $\varepsilon(H)$ of a schema H over the multary alphabet Σ. Exactly as in the binary case ([6]), we want these $\varepsilon(H)$ to satisfy the *general partition equation*

$$f(H) = \sum_{H' \supset H} \varepsilon(H'),$$

where, $H' \supset H$ means that H and H' agree on the fixed positions of H'. For example, if $n = 3$, then $H = 1\#2 \subset H' = 1\#\#$.

As might be expected, the partition coefficient of the schema $\# \ldots \#$ should be the average of the function f. On the other hand, the average value of any schema H of order 1 can be approximated by the schema $\# \ldots \#$, and the difference between this approximation and its correct value

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will be the partition coefficient of H. Let us now consider a schema H of order 2, for example $H = \# \ldots \# ab$, with $a, b \in \Sigma = \{0, 1, \ldots, n-1\}$. We may interpret H as a combination of schemata H' of order 1 such that $H' \supset H$. So, as a first approximation, we could calculate the fitness value of H as follows:

$$
\begin{aligned}
f(H) \approx{} & f(\# \ldots \#) + (f(\# \ldots \#) - f(\# \ldots a\#)) \\
& + (f(\# \ldots \#) - f(\# \ldots \# b)).
\end{aligned}
$$

Obviously, in this approximation we ignore the interaction between the fixed positions in H. In fact, the difference between the actual value of $f(H)$ and the above approximation is exactly the partition coefficient of $H = \# \ldots \# ab$:

$$
\begin{aligned}
f(\# \ldots \# ab) ={} & \varepsilon(\# \ldots \#) + \varepsilon(\# \ldots a\#) \\
& + \varepsilon(\# \ldots \# b) + \varepsilon(\# \ldots \# ab).
\end{aligned}
$$

Since $\varepsilon(\# \ldots \#) = f(\# \ldots \#)$, iterating the process and rearranging the terms in the general partition equation yields, for any schema H, that

$$
\varepsilon(H) = f(H) - \sum_{H' \supsetneq H} \varepsilon(H'),
$$

so the $\varepsilon(H)$ may be calculated recursively. For example:

$$
\varepsilon(\# \ldots a\#) = f(\# \ldots a\#) - \varepsilon(\# \ldots \#).
$$

In order to recover the original fitness function from the partition coefficients, we will need the following two results of Mason (see [12] for details):

THEOREM 2. *For any $H = h_{\ell-1} \ldots h_{i+1} \# h_{i-1} \ldots h_0$ (with $h_j \in \Sigma \cup \{\#\}$),*

$$
\sum_{a \in \Sigma} \varepsilon(h_{\ell-1} \ldots h_{i+1} a h_{i-1} \ldots h_0) = 0.
$$

THEOREM 3. *If some $\varepsilon(\cdot)$ satisfies theorem 2, then the general partition equation is satisfied for the fitness values*

$$
f(s) = \sum_{s \in H} \varepsilon(H).
$$

Theorem 2 shows that the partition coefficients are not independent of each other. We will consider a minimal collection of partition coefficients which generates the set of all partition coefficients and which, by theorem 3, also permits to recover the original fitness function. This minimal set will consist of the so-called *generalized natural Walsh coefficients*.

We proceed as follows. Denote by Σ' the alphabet Σ augmented with the "don't care" symbol $\#$ and consider the function $\widehat{\beta}$ which associates with any schema $H = h_{\ell-1} \ldots h_0 \in (\Sigma' \setminus \{0\})^\ell$ the string $\widehat{\beta}(H) = \widehat{\beta}_{\ell-1} \ldots \widehat{\beta}_0 \in \Sigma^\ell$, with

$$
\widehat{\beta}_i = \begin{cases} 0 & \text{if } h_i = \#, \\ h_i & \text{otherwise.} \end{cases}
$$

With each of these n^ℓ schemata H or, equivalently, the corresponding values $\widehat{\beta}(H)$, we associate a *generalized natural Walsh coefficient*

$$
v_H^{nat} = v_{\widehat{\beta}(H)}^{nat} = \varepsilon(H).
$$

For example, if we consider the schema $H = 12\#$ over a ternary alphabet, then

$$
v_H^{nat} = v_{\widehat{\beta}(12\#)}^{nat} = v_{15}^{nat}.
$$

If H is any schema, we may now define recursively (in the order of H), the value v_H^{nat} by

$$
v_H^{nat} = \begin{cases} (-1)^{o(H)} v_{\widehat{\beta}(H)}^{nat} & \text{if } h_i \in \Sigma' \setminus \{0\} \\ \displaystyle\sum_{a \in \{1, \ldots, n-1\}} -v_{h_{\ell-1} \ldots h_{i+1} a h_{i-1} \ldots h_0}^{nat} & \text{if } h_i = 0. \end{cases}
$$

Now, if H is any schema, the value $\varepsilon(H)$ is given by

$$
\varepsilon(H) = v_H^{nat}.
$$

The function v^{nat} satisfies the condition of theorem 2 and, by theorem 3, the general partition equation holds, i.e.,

$$
f(s) = \sum_{s \in H} v_H^{nat}. \tag{2}
$$

So, it thus follows, exactly as in (1) that the original function f may be reconstructed when these general Walsh coefficients v_H^{nat} are known. Now, we can define, for any schema $H \in \Sigma^\ell = \Omega_n$, a new schema $\beta(H)$ as follows:

$$
\beta(H)_i = \begin{cases} 0 & \text{if } h_i = \#, \\ h_i & \text{otherwise.} \end{cases}
$$

It is obvious that β and $\widehat{\beta}$ are the same if $H = h_{\ell-1} \ldots h_0 \in (\Sigma' \setminus \{0\})^\ell$.

Let us give some examples. For $n = 3$ and $\ell = 1$ the connection between the schemata and the general Walsh coefficients is given by

H	$\beta(H)$	v_H^{nat}
$\#$	0	v_0^{nat}
0	0	$v_1^{nat} + v_2^{nat}$
1	1	$-v_1^{nat}$
2	2	$-v_2^{nat}$

and for $n = 3$ and $\ell = 2$

H	$\beta(H)$	v_H^{nat}
$\#\#$	00	v_0^{nat}
$\#0$	00	$v_1^{nat} + v_2^{nat}$
$\#1$	01	$-v_1^{nat}$
$\#2$	02	$-v_2^{nat}$
$0\#$	00	$v_3^{nat} + v_6^{nat}$
00	00	$v_4^{nat} + v_5^{nat} + v_7^{nat} + v_8^{nat}$
01	01	$-v_4^{nat} - v_7^{nat}$
02	02	$-v_5^{nat} - v_8^{nat}$
$1\#$	10	$-v_3^{nat}$
10	10	$-v_4^{nat} - v_5^{nat}$
11	11	v_4^{nat}
12	12	v_5^{nat}
$2\#$	20	$-v_6^{nat}$
20	20	$-v_7^{nat} - v_8^{nat}$
21	21	v_7^{nat}
22	22	v_8^{nat}

An explicit relation between both generalized Walsh coefficients (v_H^{nat} and v_t^c) can be obtained making use of basic techniques of Linear Algebra (see [8] for details).

4. SCHEMA AVERAGES AND PARTITION COEFFICIENTS - MULTARY CASE

Let us first derive the analogues of Theorem 1 and Corollary 1 when we restrict to a schema H in Ω_n. We define, for

any schema H in Ω_n,

$$J(H) = \{t \in \Omega_n; h_i = \# \Rightarrow t_i = 0\}.$$

Then, we can prove:

LEMMA 1. *For any schema H in Ω_n, for any $s \in H$ and any $t \in J(H)$, we have*

$$\psi_t(s) = \psi_t(\beta(H)).$$

PROOF.

$$\psi_t(s) = \prod_{i=0}^{\ell-1} r^{s_i t_i} = \prod_{i;h_i=\#} r^{s_i t_i} \prod_{i;h_i \neq \#} r^{s_i t_i}.$$

As $t \in J(H)$, we have $t_i = 0$ if $h_i = \#$ and $\prod_{i;h_i=\#} r^{s_i t_i} = 1$. On the other hand, as $s \in H$ we have $s_i = h_i = \beta(H)_i$, for all i with $h_i \neq \#$. We thus obtain

$$\psi_t(s) = \prod_{i;h_i \neq \#} r^{s_i t_i} = \psi_t(\beta(H)),$$

as claimed. $\square$

THEOREM 4 (SECOND BALANCED SUM THEOREM). *For any $t \in \Omega_n$ and any schema H over Ω_n, we have*

$$\sum_{s \in H} \psi_t(s) = \begin{cases} |H|\psi_t(\beta(H)) & \text{if } t \in J(H) \\ 0 & \text{if } t \notin J(H). \end{cases}$$

PROOF. If $t \in J(H)$, the assertion follows from lemma 1. If $t \notin J(H)$, then there exists at least, one non-zero component of t, say t_i, with $h_i = \#$. Let us suppose that $i = \ell - 1$, then, arguing as in Theorem 1, it follows that $\sum_{s \in H} \psi_t(s) = 0$, indeed. $\square$

Also, we can obtain:

COROLLARY 2. *For any schema H over Ω_n, we have*

$$f(H) = \sum_{t \in J(H)} v_t^c \psi_t(\beta(H)).$$

In particular,

$$f(\Omega_n) = f(\# \ldots \#) = n^{-\ell/2} v_0^c.$$

EXAMPLE 1. *Let us include some examples of the relation between f and its general Walsh coefficients.*

Let H_a be the schema $H_a = \# \ldots \# a \# \ldots \#$ with $a \in \Sigma$ at position i. Then $\beta(H_a) = 0 \ldots 0\, a\, 0 \ldots 0$ and $J(H_a) = 0 \cdots 0 \# 0 \cdots 0 = \{kn^i;\ 0 \leq k < n\}$. So, for $t = kn^i \in J(H_a)$,

$$\psi_t(\beta(H_a)) = r^{k \cdot a}$$

and

$$\begin{aligned} f(H_a) &= n^{-\ell/2} \sum_{t \in J(H_a)} w_t \psi_t(\beta(H_a)) \\ &= n^{-\ell/2} \left(w_0 + r^a \cdot w_{n^i} + \cdots + r^{(n-1)a} \cdot w_{(n-1)n^i} \right), \end{aligned}$$

i.e.,

$$f(H_a) = n^{-\ell/2} \sum_{k=0}^{(n-1)} r^{k \cdot a} \cdot w_{k \cdot n^i}.$$

Now, consider the order 2 schema

$$H_{ab} = \# \ldots \# \overset{\overset{i}{\downarrow}}{a} \# \ldots \# \overset{\overset{j}{\downarrow}}{b} \# \cdots \#,$$

with $a, b \in \Sigma$.
In this case,

$$\beta(H_{ab}) = 0 \cdots 0 \overset{\overset{i}{\downarrow}}{a} 0 \cdots 0 \overset{\overset{j}{\downarrow}}{b} 0 \cdots 0$$

and

$$\begin{aligned} J(H_{ab}) &= 0 \cdots 0 \overset{\overset{j}{\downarrow}}{\#} 0 \cdots 0 \overset{\overset{j}{\downarrow}}{\#} 0 \cdots 0 \\ &= \{pn^i + qn^j;\ 0 \leq p, q < n\} \end{aligned}$$

So, for all $t \in J(H_{ab})$

$$\psi_t(\beta(H_{ab})) = r^{at_i + bt_j} = r^{ap+bq},$$

for some $0 \leq p, q < n$. We thus obtain:

$$\begin{aligned} f(H_{ab}) &= n^{-\ell/2} \sum_{t \in J(H_{ab})} w_t \psi_t(\beta(H_{ab})) \\ &= n^{-\ell/2} \sum_{p,q=0}^{(n-1)} r^{ap+bq} \cdot w_{p \cdot n^i + q \cdot n^j}. \end{aligned}$$

Finally, we want to establish a link between partition coefficients and the general Walsh coefficients (v_t^c) defined in the first section. As there is no possible ambiguity, we will use the notation v_t when we refer to these Walsh coefficients.

We start by defining

$$\alpha(H) = \{t \in \Omega_n; h_i = \# \Leftrightarrow t_i = 0\}.$$

We will need the following technical lemmas.

LEMMA 2. *For any pair of schemata $H, H' \in \Sigma^\ell$, we have*

1. *if $H' \supset H$ then $\alpha(H') \subset J(H)$;*

2. *for any $t \in J(H)$, there exists exactly one $H' \supset H$ with $t \in \alpha(H')$.*

PROOF. ¿From the definition of α it follows that

$$t \in \alpha(H') \Leftrightarrow \left(h_i' = \# \Leftrightarrow t_i = 0, \text{for all } 0 \leq i < \ell \right).$$

So, as $H' \supset H$, if $h_i = \#$ then $h_i' = \#$ and we have that $t_i = 0$. This implies that $t \in J(H)$.

To prove the existence of an H' for the second statement, we define $H' = h'_{\ell-1} \cdots h'_0$ by

$$h_i' = \begin{cases} \# & t_i = 0 \\ h_i & t_i \neq 0. \end{cases}$$

Obviously, $H' \supset H$. On the other hand, as $t \in J(H)$, then $t \in \alpha(H')$. Indeed, if $t_i \neq 0$ then we have that $h_i' \neq \#$ and so $h_i' = h_i \neq \#$. The fact that $h_i' = \#$, when $t_i = 0$ follows directly from the definition of H'.

Moreover, H' is unique as such. In fact, if $t_i = 0$ then $h_i' = \#$, as $t \in \alpha(H')$.

If $t_i \neq 0$ then, $h_i \neq \#$ (as $t \in J(H)$) and necessarily $h_i' \neq \#$ (because $h_i' = \#$ would imply $t_i = 0$). So, as $H' \supset H$, it follows that $h_i' = h_i$. $\square$

LEMMA 3. *For any schema H over a multary alphabet, we have*

$$J(H) = \bigcup_{H' \supset H} \alpha(H') \quad (\text{disjoint union}).$$

PROOF. Note that it only remains to prove that for any pair of schemata $H', H'' \supset H$ we have $H' \neq H'' \Leftrightarrow \alpha(H') \cap \alpha(H'') = \emptyset$. If $\alpha(H') \cap \alpha(H'') \neq \emptyset$ then, by picking $t \in \alpha(H') \cap \alpha(H'')$, we have that $t \in J(H)$ and the unicity given in lemma 2 implies $H' = H''$.

The fact that $H' = H''$ implies $\alpha(H') \cap \alpha(H'') \neq \emptyset$ is obvious. $\square$

LEMMA 4. *For any $H' \supset H$ and for any $t \in \alpha(H')$ we have $\psi_t(\beta(H)) = \psi_t(\beta(H'))$.*

PROOF. Let us check that $t \cdot \beta(H) = t \cdot \beta(H')$. It suffices to verify for all i that $t_i \cdot \beta_i = t_i \cdot \beta_i'$, when we write β_i (resp. β_i') for $\beta(H)_i$ (resp. $\beta(H')_i$).

If $h_i = \#$ then $\beta_i = 0$ and also $h_i' = \#$, so $\beta_i' = 0$.

If $h_i \neq \#$, as $H' \supset H$ we need to distinguish two cases:

1. if $h_i' = h_i$ then $\beta_i = h_i = h_i' = \beta_i'$;

2. if $h_i' = \#$ then $t_i = 0$ (as $t \in \alpha(H')$).

This proves the assertion. $\square$

Finally, we reach our aim with the following theorem which states that the partition coefficients of any schema H may be computed using only these Walsh coefficients v_t for which the number of loci with value different from zero in t is at most equal to the order of H.

THEOREM 5. *For any schema H over Ω_n, we have*

$$\varepsilon(H) = \sum_{t \in \alpha(H)} v_t \psi_t(\beta(H)).$$

PROOF. First note that

$$\begin{aligned} f(H) &= \sum_{t \in J(H)} v_t \psi_t(\beta(H)) \\ &= \sum_{H' \supset H} \sum_{t \in \alpha(H')} v_t \psi_t(\beta(H)) \\ &= \sum_{H' \supset H} \sum_{t \in \alpha(H')} v_t \psi_t(\beta(H')). \end{aligned}$$

Since the result is true for $H = \# \ldots \#$ (as $\alpha(H) = \alpha(\Omega_n) = \{0 \cdots 0\} = \beta(\Omega_n)$, $\psi_0(\beta(H)) = 1$ and so $f(\Omega_n) = v_0$), we may argue by induction in the order of H.

Let us assume that

$$\varepsilon(H') = \sum_{t \in \alpha(H')} v_t \psi_t(\beta(H'))$$

for all $H' \supsetneq H$. We then have

$$\begin{aligned} \varepsilon(H) &= f(H) - \sum_{H' \supsetneq H} \varepsilon(H') \\ &= \sum_{H' \supset H} \sum_{t \in \alpha(H')} v_t \psi_t(\beta(H')) - \sum_{H' \supsetneq H} \sum_{t \in \alpha(H')} v_t \psi_t(\beta(H')) \\ &= \sum_{t \in \alpha(H)} v_t \psi_t(\beta(H)). \end{aligned}$$

$\square$

EXAMPLE 2. *To conclude, let us include some easy examples of the previous relation. If we denote, as before, by H_a*

the order 1 *schema* $H_a = \# \cdots \# \overset{\overset{i}{\downarrow}}{a} \# \cdots \#$, *with $0 \leq a < n$, we already know that*

$$\beta(H_a) = 0 \cdots 0 \overset{\overset{i}{\downarrow}}{a} 0 \cdots 0,$$

and also that $\psi_t(\beta(H_a)) = r^{a \cdot t_i}$. As it can easily be seen that

$$\begin{aligned} \alpha(H_a) &= \{t = 0 \cdots 0 t_i 0 \cdots 0; \, t_i \neq 0\} \\ &= \{kn^i; \, 0 < k < n\}, \end{aligned}$$

we obtain:

$$\varepsilon(H_a) = \sum_{t \in \alpha(H_a)} r^{a \cdot t_i} v_t = \sum_{k=1}^{n-1} r^{a \cdot k} v_{k \cdot n^i}.$$

For example, for ternary alphabets ($n = 3$), we have $r = -\frac{1}{2} + i \frac{\sqrt{3}}{2}$ and

$$\begin{aligned} \varepsilon(\# \cdots \# \overset{\overset{i}{\downarrow}}{0} \# \cdots \#) &= v_{3^i} + v_{2 \cdot 3^i}, \\ \varepsilon(\# \cdots \# \overset{\overset{i}{\downarrow}}{1} \# \cdots \#) &= r v_{3^i} + r^2 v_{2 \cdot 3^i}, \\ \varepsilon(\# \cdots \# \overset{\overset{i}{\downarrow}}{2} \# \cdots \#) &= r^2 v_{3^i} + r v_{2 \cdot 3^i}. \end{aligned}$$

Finally, let us consider again the order 2 schemata

$$H_{ab} = \# \cdots \# \overset{\overset{i}{\downarrow}}{a} \# \cdots \# \cdots \# \overset{\overset{j}{\downarrow}}{b} \# \cdots \#,$$

with $a, b \in \{0, 1, \ldots, n-1\}$ Then, we already know that

$$\beta(H_{ab}) = 0 \cdots 0 \overset{\overset{i}{\downarrow}}{a} 0 \cdots 0 \overset{\overset{j}{\downarrow}}{b} 0 \cdots 0,$$

and $\psi_t(\beta(H_{ab})) = r^{a \cdot p + b \cdot q}$ for $t \in \Omega_n$. Taking into account that

$$\begin{aligned} \alpha(H_{ab}) &= \{t = 0 \cdots 0 t_i 0 \cdots 0 \cdots 0 t_j 0 \cdots 0; \, t_i, t_j \neq 0\} \\ &= \{pn^i + qn^j; \, 0 < p, q < n\} \end{aligned}$$

we have

$$\begin{aligned} \varepsilon(H_{ab}) &= n^{-\ell/2} \sum_{t \in \alpha(H_{ab})} r^{a \cdot t_i + b \cdot t_j} w_t \\ &= n^{-\ell/2} \sum_{p,q=1}^{n-1} r^{a \cdot p + b \cdot q} w_{pn^i + qn^j}. \end{aligned}$$

For example, if $n = 3$ we obtain

$$
\begin{aligned}
\varepsilon(H_{00}) &= \varepsilon(\#\cdots\#\overset{\underset{\downarrow}{i}}{0}\#\cdots\#\cdots\#\overset{\underset{\downarrow}{j}}{1}\#\cdots\#) \\
&= 3^{-\ell/2} \sum_{p,q=1}^{2} w_{p\cdot 3^i + q\cdot 3^j} \\
\varepsilon(H_{01}) &= \varepsilon(\#\cdots\#\overset{\underset{\downarrow}{i}}{0}\#\cdots\#\cdots\#\overset{\underset{\downarrow}{j}}{1}\#\cdots\#) \\
&= 3^{-\ell/2} \sum_{p,q=1}^{2} r^q w_{p\cdot 3^i + q\cdot 3^j} \\
\varepsilon(H_{21}) &= \varepsilon(\#\cdots\#\overset{\underset{\downarrow}{i}}{2}\#\cdots\#\cdots\#\overset{\underset{\downarrow}{j}}{1}\#\cdots\#) \\
&= 3^{-\ell/2} \sum_{p,q=1}^{2} r^{2\cdot p + q} w_{p\cdot 3^i + q\cdot 3^j},
\end{aligned}
$$

for instance.

5. CONCLUDING REMARKS

The main purpose of this note was to generalize the machinery of Walsh transforms from binary to multary alphabets, with emphasis on their utility in the calculation of partition coefficients. We included some concrete examples showing the strength, in the multary case as well, of the use of Walsh coefficients in this context. As a nice side-effect, it appeared that our techniques not only provide for a nice extension of "classical" results to the multary case, but also yields shorter and more transparent proofs of the latter.

6. ACKNOWLEDGMENTS

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