Exact Analysis of the Sampling Distribution for the Canonical Particle Swarm Optimiser and its Convergence during Stagnation

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ABSTRACT

Several theoretical analyses of the dynamics of particle swarms have been offered in the literature over the last decade. Virtually all rely on substantial simplifications, including the assumption that the particles are deterministic. This has prevented the exact characterisation of the sampling distribution of the PSO. In this paper we introduce a novel method, which allows one to exactly determine all the characteristics of a PSO's sampling distribution and explain how they change over any number of generations, in the presence stochasticity. The only assumption we make is stagnation, i.e., we study the sampling distribution produced by particles in search for a better personal best. We apply the analysis to the PSO with inertia weight, but the analysis is also valid for the PSO with constriction.

Categories and Subject Descriptors

I.2.8 [**Artificial Intelligence**]: Problem Solving, Control Methods, and Search

General Terms

Performance

Keywords

Particle Swarm Optimisation, Theory

Algorithm 1 Classical PSO.

```
Initialize a population array of particles with random
  positions and velocities on D dimensions in the problem
  space.
loop
  For each particle, evaluate the desired optimization
    fitness function in D variables.
  Compare particle's fitness evaluation with its personal
    best fitness pbest^i. If current value is better than
    pbest^i, then set pbest^i equal to the current value, and
    y^i equal to the current location x^i in D-dimensional
    space.
  Identify the particle in the neighbourhood with the
    best success so far, and assign its position to the vari-
    able ŷ.
  Change the velocity and position of the particle ac-
    cording to the following equations:
```

$$v_{t+1}^i = wv_t^i + \phi_1 \otimes (y^i - x_t^i) + \phi_2 \otimes (\hat{y} - x_t^i) \quad (1)$$

$$x_{t+1}^i = x_t^i + v_{t+1}^i \quad (2)$$

```
  If a criterion is met, exit loop.
end loop
```

Note: ϕ_i represents a vector of random numbers uniformly distributed in $[0, c_i]$ and $\otimes$ is component-wise multiplication.

1. INTRODUCTION

We consider the basic form of PSO with inertia weight shown in Algorithm 1. Despite its apparent simplicity, this PSO has presented formidable challenges to those interested in swarm intelligence theory. Firstly, the PSO is made up of a large number of interacting elements (the particles). Although the nature of the elements and of the interactions is simple, understanding the dynamics of the whole is nontrivial. Secondly, the particles are provided with memory and (albeit limited) intelligence, which mean that from one iteration to the next a particle may be attracted towards a new y_i or a new $\hat{y}$ or both. Thirdly, forces are stochastic. This prevents the use of standard mathematical tools used in the analysis of deterministic dynamical systems. Fourthly, the behaviour of the PSO depends crucially on the structure of the fitness function. However, PSOs have been used on such a wide range of fitness functions that it is difficult to characterise a useful function space in which to study the role of the fitness function, and so it is hard to find general results. Nonetheless some progress has been made, by considering simplifying assumptions such as isolated single individuals, search stagnation (i.e., no improved solutions are found) and, crucially, *absence of randomness*.

For example, Ozcan and Mohan [3] studied the behaviour of one particle, in isolation, in one dimension, in the absence of stochasticity and during stagnation. Also, y and $\hat{y}$ were assumed to coincide, as is the case for the best particle in a neighbourhood. The work was extended in [4] where mul-

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tiple multi-dimensional particles were covered. Similar assumptions were used by Clerc and Kennedy's model [5]: one particle, one dimension, deterministic behaviour and stagnation. Under these conditions the swarm is a discrete-time linear dynamical system. The dynamics of the state (position and velocity) of particle can be determined by finding the eigenvalues and eigenvectors of the state transition matrix. The model, therefore, predicts that the particle will converge to equilibrium if the magnitude of the eigenvalues is smaller than 1.

A similar approach was used by van den Bergh [6] (see also [2]), who, again, modelled one particle, with no randomness and during stagnation. As in previous work, van den Bergh provided an explicit solution for the trajectory of the particle. He showed that the particle is attracted towards a fixed point. He also argued that the analysis would be valid also in the presence of stochasticity. [6] also suggested the possibility that particles may converge on a point that is neither the global optimum nor indeed a local optimum. This implies that a PSO is not guaranteed to be an optimiser.

A simplified model of particle was also studied by Yasuda *et al.* [7]. The assumptions were: one one-dimensional particle, stagnation and absence of stochasticity. Inertia was included in the model. Again an eigenvalue analysis of the resulting dynamical system was performed with the aim of determining for what parameter settings the systems is stable and what classes of behaviours are possible for a particle. Conditions for cyclic behaviour were analysed in detail.

Blackwell [19] investigated how the spatial extent of a particle swarm varies over time. A simplified swarm model was adopted which is an extension of the one by Clerc and Kennedy where more than one particle and more than one dimensions are allowed. This allowed particles to interact, in the sense that they could change their personal best. Constriction was included but not stochasticity. [19] suggested that spatial extent decreases exponentially with time.

Brandstätter and Baumgartner [10] drew an analogy between Clerc and Kennedy's model [5] and a damped mass-spring oscillator, making it possible to rewrite the model using the notions of damping factor and natural vibrational frequency. Like the original model, this model assumes one particle, one dimension, no randomness and stagnation.

Under the same assumptions as [5] and following a similar approach, Trelea [11] performed a lucid analysis of a 4-parameter family of particle models and identified regions in the parameter space where the model exhibits qualitatively different behaviours (either stability, harmonic oscillations or zigzagging behaviour).

The dynamical system approach proposed by Clerc and Kennedy has recently been extended by Campana *et al.* [12, 13] who studied an extended PSO. Under the assumption that no randomness is present, the resulting model is a discrete, linear and stationary dynamical system, for which [12, 13] formally expressed the free and forced responses. However, since the forced response depends inextricably on the specific details of the fitness function, they were able to study in detail only the free response.

To better understand the behaviour of the PSO during phases of stagnation, Clerc [16] analysed the distribution of velocities of one particle controlled by the standard PSO update rule with inertia and *stochastic forces*. In particular, he was able to show that a particle's new velocity is the sum of three components: a forward force, a backward force and noise. Clerc studied the distributions of these forces.

Kadirkamanathan *et al.* [17] were able to study the stability of particles *in the presence of stochasticity* by using Lyapunov stability analysis. They considered the behaviour of a single particle – the swarm best – with inertia and during stagnation. By representing the particle as a non-linear feedback system, they were able to apply a large body of knowledge from control theory. E.g., they found sufficient conditions on the PSO parameters to guarantee convergence. Since Lyapunov theory is very conservative, the conditions found are very restrictive, effectively forcing the PSO to have little oscillatory behaviour.

In summary, with very few exceptions all mathematical models of PSO behaviour have been obtained under rather unrealistic assumptions. In particular, very little is known regarding how the sampling distribution of particles changes over time. In this paper we introduce a novel method, which allows one to exactly determine all the characteristics of a PSO's sampling distribution and explain how they change over any number of generations. The only assumption we make is stagnation, i.e., we study the sampling distribution produced by particles in search for a better personal best.

We will apply the analysis to the PSO with inertia weight (Algorithm 1). However, we should note that a PSO with constriction (see[5]) is algebraically equivalent to a PSO with inertia. Indeed, in this PSO, particles are controlled by the equation

$$v_{t+1}^i = \chi\left(v_t^i + \tilde{\phi}_1 \otimes (y^i - x_t^i) + \tilde{\phi}_2 \otimes (\hat{y} - x_t^i)\right) \quad (3)$$

which can be transformed into one Equation (1) via the mapping $\chi \to w$ and $\chi\tilde{\phi}_i \to \phi_i$. So, the theory developed in the rest of this paper will apply to the PSO with constriction as well.

The paper is organised as follows. In Section 2 we derive recursions for the dynamics of first and second order statistics of the sampling distribution of a PSO's particle during stagnation. We study the fixed-points for these quantities and stability in Section 3. In Section 4 we show the results of numerically integrating the dynamic equations for the distribution's statistics. Finally, we provide some discussion, indications for future work and our conclusions in Section 5.

2. DYNAMICS OF FIRST AND SECOND MOMENTS OF THE PSO SAMPLING DISTRIBUTION

If the PSO in a stagnation phase (i.e., there are no fitness improvements), each particle effectively behaves independently. Also, each dimension is treated independently. So, we can analyse each particle's behaviour in isolation. Droping the superscript i in Equations (1) and (2), we can rewrite them as a single (second order) difference equations, as was done other researchers (e.g., in [6]), by making use of the fact that $v_t = x_t - x_{t-1}$. We obtain

$$x_{t+1} = x_t(1 + w) - x_t(\phi_1 + \phi_2) - wx_{t-1} + \phi_1 y + \phi_2 \hat{y}. \quad (4)$$

2.1 Dynamics of $E[x_t]$

Unlike previous research, we will not make the simplifying assumption that ϕ_1 and ϕ_2 are constant in Equation (9). Instead, we treat them for what they are, i.e., uniformly distributed stochastic variables, and we apply the expectation operator to both sides of the equation obtaining

$$\begin{aligned} E[x_{t+1}] &= E[x_t](1+w) - E[x_t](E[\phi_1] + E[\phi_2]) \\ &- wE[x_{t-1}] + E[\phi_1]y + E[\phi_2]\hat{y} \end{aligned} \tag{5}$$

where we performed the substitution $E[x_t\phi_i] = E[x_t]E[\phi_i]$ because of the statistical independence between ϕ_i and x_t.[1] Because ϕ_i is uniformly distributed in $[0, c_i]$ we have

$$E[\phi_1] = \frac{c_1}{2} \qquad E[\phi_2] = \frac{c_2}{2} \tag{6}$$

and, so,

$$E[x_{t+1}] = E[x_t](1+w-\frac{c_1+c_2}{2})-wE[x_{t-1}]+\frac{c_1}{2}y+\frac{c_2}{2}\hat{y} \tag{7}$$

Let p be a fixed point for this equation. This requires

$$p = \frac{c_1 y + c_2 \hat{y}}{c_1 + c_2} \tag{8}$$

For the sake of simplicity let us now restrict our attention to the case $c_1 = c_2 = c$. Furthermore, let us rename $(1 + w) = w'$. So

$$x_{t+1} = x_t w' - x_t\phi_1 - x_t\phi_2 - wx_{t-1} + \phi_1 y + \phi_2\hat{y} \tag{9}$$

and

$$E[x_{t+1}] = E[x_t](w' - c) - wE[x_{t-1}] + c\frac{y+\hat{y}}{2} \tag{10}$$

Naturally, the stability of this equation is determined by the magnitude of the roots of the associated characteristic polynomial, or of the eigenvalues of the associated first-order vectorial difference equation. Figure 1 plots the magnitude of the largest eigenvalue of the equation for $c = 0.01, 0.02, ..., 4.00$ and $w = 0.01, 0.02, ..., 1.0$. The line on the surface encloses the stable region.

Note that if we assumed that that ϕ_1 and ϕ_2 are constant and equal to their maximum value, c, Equation (9) would become

$$x_{t+1} = x_t(w' - 2c) - wx_{t-1} + c(y + \hat{y}) \tag{11}$$

This equation has been studied extensively in previous research and has exactly the same form as Equation (10), except that here we have $2c$ instead of c and the magnitude of the forcing term, $c(y + \hat{y})$, is doubled. So, the stability of Equation (10) has effectively been studied in previous research (e.g., [11], [6] and [5]; see also [2] for an extensive review). Indeed, the stable region depicted in Figure 1 is exactly the same as reported in [11, Figure 1(a)], and the explicit dynamics of $E[x_t]$ is explicitly given in previous work (e.g., [6]) if parameters are appropriately rescaled.

[1]Note, ϕ_i are stochastic variables sampled at iteration t. At that iteration they are independent of x_t. They are not independent of x_{t+1}, x_{t+2}, etc.

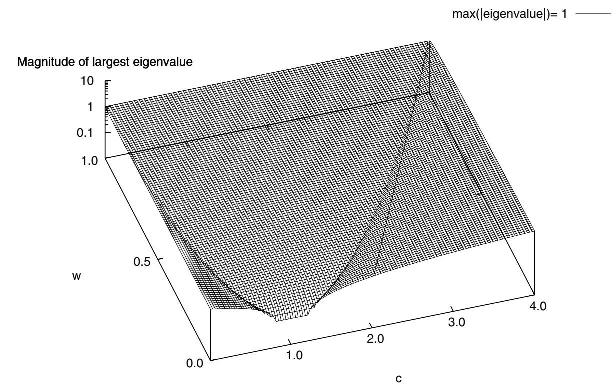

Figure 1: Stability analysis based on the difference equation for $E[x_t]$ as a function of the parameters w and c. The line on the surface encloses the presumed stable region.

2.2 Dynamics of $E[x_t^2]$, $E[x_t x_{t-1}]$ and $StdDev[x_t]$

Let us now compute x_{t+1}^2:

$$\begin{aligned} x_{t+1}^2 &= (x_t w' - x_t\phi_1 - x_t\phi_2 - wx_{t-1} + \phi_1 y + \phi_2\hat{y})^2 \\ &= x_t^2 w'^2 - x_t^2\phi_1 w' - x_t^2\phi_2 w' - wx_{t-1}x_t w' + \phi_1 y x_t w' \\ &+ \phi_2\hat{y}x_t w' - x_t^2 w'\phi_1 + x_t^2\phi_1^2 + x_t^2\phi_2\phi_1 + wx_{t-1}x_t\phi_1 \\ &- \phi_1^2 y x_t - \phi_2\hat{y}x_t\phi_1 - x_t^2 w'\phi_2 + x_t^2\phi_1\phi_2 + x_t^2\phi_2^2 \\ &+ wx_{t-1}x_t\phi_2 - \phi_1 y x_t\phi_2 - \phi_2^2\hat{y}x_t - x_t w' w x_{t-1} \\ &+ x_t\phi_1 w x_{t-1} + x_t\phi_2 w x_{t-1} + w^2 x_{t-1}^2 - \phi_1 y w x_{t-1} \\ &- \phi_2\hat{y}wx_{t-1} + x_t w'\phi_1 y - x_t\phi_1^2 y - x_t\phi_2\phi_1 y - wx_{t-1}\phi_1 y \\ &+ \phi_1^2 y^2 + \phi_2\hat{y}y\phi_1 + x_t w'\phi_2\hat{y} - x_t\phi_1\phi_2\hat{y} - x_t\phi_2^2\hat{y} \\ &- wx_{t-1}\phi_2\hat{y} + \phi_1 y\phi_2\hat{y} + \phi_2^2\hat{y}^2 \end{aligned}$$

Again we apply the expectation operator to both sides of the equation, obtaining

$$\begin{aligned} &E[x_{t+1}^2] \\ &= E[x_t^2]\left(w'^2 - 4\mu w' + 2\nu + 2\mu^2\right) \\ &+ E[x_{t-1}x_t]\left(-2ww' + 4w\mu\right) \\ &+ E[x_{t-1}^2]\left(w^2\right) \\ &+ E[x_t]\left(2\mu y w' + 2\mu\hat{y}w' - 2\nu y - 2\mu^2\hat{y} - 2\mu^2 y - 2\nu\hat{y}\right) \\ &+ E[x_{t-1}]\left(-2\mu y w - 2\mu\hat{y}w\right) \\ &+ \nu y^2 + 2\mu^2 y\hat{y} + \nu\hat{y}^2 \end{aligned}$$

where we set $\mu = E[\phi_i] = c/2$ and $\nu = E[\phi_i^2] = c^2/3$, for brevity.

As we discussed in Section 2.1, we have a recursion (and in fact an explicit solution) for $E[x_t]$, so the recursion in Equation (17) could be solved if we had a recursion for $E[x_t x_{t-1}]$. Let us obtain such a recursion.

We multiply both sides of Equation (9) by x_t, obtaining

$$x_{t+1}x_t = x_t^2 w' - x_t^2(\phi_1+\phi_2) - wx_t x_{t-1} + x_t\phi_1 y + x_t\phi_2\hat{y} \tag{12}$$

thereby

$$E[x_{t+1}x_t] = E[x_t^2](w' - c) - wE[x_t x_{t-1}] + E[x_t]c\frac{y+\hat{y}}{2} \tag{13}$$

With this additional equation we are now in a position to determine the dynamics of $E[x_t^2]$ and $E[x_t x_{t-1}]$, in addition to the dynamics of $E[x_t]$ we derived in Section 2.1. Then, by using the relation

$$StdDev[x_t] = \sqrt{E[x_t^2] - (E[x_t])^2} \tag{14}$$

one can derive the dynamics for the standard deviation of the sampling distribution of a PSO during stagnation.

2.3 Initial conditions

The recursions for $E[x_t]$, $E[x_t^2]$ and $E[x_t x_{t-1}]$ form the following set of coupled difference equations

$$\left\{\begin{array}{rcl} E[x_{t+1}] &=& E[x_t](w'-c) - wE[x_{t-1}] + c\frac{y+\hat{y}}{2} \\ E[x_{t+1}^2] &=& E[x_t^2]\left(w'^2 - 4\mu w' + 2\nu + 2\mu^2\right) + \\ && E[x_{t-1}x_t]\left(-2ww' + 4w\mu\right) + \\ && E[x_{t-1}^2]\left(w^2\right) + \\ && 2E[x_t](y+\hat{y})\left(\mu w' - \nu - \mu^2 y\right) - \\ && 2w\mu E[x_{t-1}](y+\hat{y}) + \nu y^2 + 2\mu^2 y\hat{y} + \nu\hat{y}^2 \\ E[x_{t+1}x_t] &=& E[x_t^2](w'-c) - wE[x_t x_{t-1}] + E[x_t]c\frac{y+\hat{y}}{2} \end{array}\right. \tag{15}$$

Let us evaluate the initial conditions for this system. To do so, we must specify how we perform the initialisation of the particle swarm. As an example, let us consider the following very typical conditions: a) a particle's initial position, x_0, is chosen uniformly at random in a symmetric range $[-\Omega, \Omega]$, b) a particle's initial velocity, v_0, is also chosen uniformly at random in the same range.

In these conditions, clearly, $E[x_0] = 0$ and $E[v_0] = 0$. So, $E[x_1] = E[x_0 + v_1] = E[x_0] + E[v_1] = E[v_1]$. Let us compute $E[v_1]$. We have that

$$\begin{array}{rcl} E[v_1] &=& E[wv_0 + \phi_1(y - x_0) + \phi_2(\hat{y} - x_0)] \\ &=& wE[v_0] + E[\phi_1](y - E[x_0]) + E[\phi_2](\hat{y} - E[x_0]) \\ &=& c\dfrac{y+\hat{y}}{2}. \end{array}$$

So, $E[x_1] = c\frac{y+\hat{y}}{2}$. We also have that $E[x_0^2] = E[v_0^2] = \frac{\Omega^2}{3}$, while $E[x_1 x_0] = E[(x_0 + v_1)x_0] = \frac{\Omega^2}{3} + E[v_1 x_0]$, the second term of which is given by

$$\begin{array}{rcl} E[v_1 x_0] &=& E[wv_0 x_0 + \phi_1(yx_0 - x_0^2) + \phi_2(\hat{y}x_0 - x_0^2)] \\ &=& wE[v_0]E[x_0] + E[\phi_1](yE[x_0] - E[x_0^2]) \\ &+& E[\phi_2](\hat{y}E[x_0] - E[x_0^2]) \\ &=& -cE[x_0^2] \\ &=& -c\dfrac{\Omega^2}{3}, \end{array}$$

resulting in $E[x_1 x_0] = (1-c)\frac{\Omega^2}{3}$.

The only remaining initial condition we need is $E[x_1^2] = E[(x_0+v_1)^2] = E[x_0^2] + 2E[v_1 x_0] + E[v_1^2] = (1-2c)\frac{\Omega^2}{3} + E[v_1^2]$, which, after similar additional calculations leads to $E[x_1^2] = \frac{7c^2-12c+6w^2+6}{18}\Omega^2 + c^2\frac{(y+\hat{y})^2}{3}$.

3. STABILITY ANALYSIS FOR PARTICLES WITH RANDOMNESS

The system of equations (15) can be written in matrix notation as an extended first order system obtaining

$$\mathbf{z}(t+1) = M\mathbf{z}(t) + \mathbf{b} \tag{16}$$

where

$$\mathbf{z}(t) = \left(E[x_t] \;\; E[x_{t-1}] \;\; E[x_t^2] \;\; E[x_{t-1}^2] \;\; E[x_t x_{t-1}]\right)^T$$

and the matrix M and the forcing vector $\mathbf{b}$ are given in Figure 2.

It is then trivial to verify under what conditions $E[x_t]$, $E[x_t^2]$ and $E[x_t x_{t-1}]$ (thereby also $StdDev[x_t]$) will converge to stable fixed-points. We need to have that all eigenvalues of M must be within the unit circle, i.e. $\Lambda_m = \max_i |\lambda_i| < 1$. The analysis of the stability of the system can be done easily. Any good computer algebra system can provide these eigenvalues in symbolic form. Two of them are simply:

$$\frac{1 + w - c \pm \sqrt{(w-c)^2 - 2c - 2w + 1}}{2}$$

The expressions for remaining three, however, are too big to report in this paper. The analysis reveals that none of the eigenvalues depends on either y or $\hat{y}$ (nor p). That is, whether or not the system is stable does not depend on where personal best and swarm best are located in the search space.

Naturally, when $\Lambda_m < 1$, in principle we could symbolically derive the fixed-point for the system, which we will denote as $\mathbf{z}^*$. This would be simply given by

$$\mathbf{z}^* = (I - M)^{-1}\mathbf{b}$$

For simplicity, below we will find explicit expressions for some components of $\mathbf{z}^*$ by other means.

When the system is stable, by the simple change of variables $\tilde{\mathbf{z}}(t) = \mathbf{z}(t) - \mathbf{z}^*$ can then represent the dynamics of the system via following linear homogeneous equation

$$\tilde{\mathbf{z}}(t+1) = M\tilde{\mathbf{z}}(t)$$

which can trivially be integrated to obtain the explicit solution

$$\tilde{\mathbf{z}}(t) = M^t\tilde{\mathbf{z}}(0).$$

Naturally, all these operations can be performed numerically once c and w are fixed. For example, in Figure 3 shows a plot of Λ_m as a function of c and w for $y = -1$ and $\hat{y} = 1$. The plot also shows a line where $\Lambda_m = 1$. As we explained earlier, although in order to compute M we have to specify y and $\hat{y}$ as well as c and w, Λ_m is not affected by what values y and $\hat{y}$ have. So, one obtains exactly the same plot, for example, for $y = 9$ and $\hat{y} = 10$ (same distance as $y = -1$ and $\hat{y} = 1$, but different p) or $y = -10$ and $\hat{y} = 10$ (different distance, but same p as for $y = -1$ and $\hat{y} = 1$).

Naturally, knowing the region where the system is stable allows one to perform an informed choice of the parameters of the PSO. We should note that in this respect the region of convergence provided by the analysis of the $E[x_t]$ alone, as it has effectively been done in previous research, does not provide enough information to guarantee convergence of the particles. It only guarantees convergence of the mean. Compare, for example Figure 3 with Figure 1. Note how the actual region of stability shown in Figure 3 lays completely inside the presumed region of stability obtained by analysing $E[x_t]$ only (Figure 1). Interestingly, by choosing parameters between the two curves, one obtains PSOs where $E[x_t] \to p$, but $StdDev[x_t]$ drifts (perhaps slowly) to infinity, which might be a desirable property if one wants to obtain PSOs capable of escaping from local optima. Note also, that choosing parameters c and w within the region of

$$M = \begin{pmatrix} w' - c & -w & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 4p\left(\mu w' - \nu - \mu^2\right) & -4\mu w p & w'^2 - 4\mu w' + 2\nu + 2\mu^2 & w^2 & 2w\left(2\mu - w'\right) \\ 0 & 0 & 1 & 0 & 0 \\ cp & 0 & w' - c & 0 & -w \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} cp \\ 0 \\ \nu y^2 + 2\mu^2 y\hat{y} + \nu \hat{y}^2 \\ 0 \\ 0 \end{pmatrix}$$

Figure 2: State update matrix M and forcing vector b (see Equation (16)).

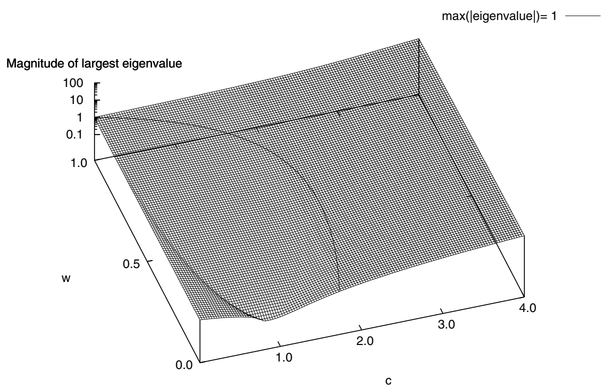

Figure 3: Magnitude of the largest eigenvalue of M as a function of the parameters w and c when $y = -1$ and $\hat{y} = 1$. The curved line on the surface encloses the stable region.

convergence *does not* imply that $StdDev[x_t] \to 0$. In the following we clarify when this is the case.

Simple inspection of the equations in Equation (15), reveals that the dynamics of $E[x_t]$ is independent from those of $E[x_t^2]$ and $E[x_t x_{t-1}]$, while the converse is not true. This means that $E[x_t^2]$ and $E[x_t x_{t-1}]$ cannot be at a fixed-point unless also $E[x_t]$ is. Let us assume that (c, w) is in the region of convergence for $E[x_t]$. Then, for sufficienlty large t, $E[x_t]$ becomes almost indistiguisheable from the fixed-point $p = \frac{y+\hat{y}}{2}$. In these conditions, for the purpose of finding fixed-points for $E[x_t^2]$ and $E[x_t x_{t-1}]$, we can replace $E[x_t]$ and $E[x_{t-1}]$ with p in the second and third equations of Equation (15), obtaining

$$\begin{aligned} E[x_{t+1}^2] &= E[x_t^2]\left(w'^2 - 4\mu w' + 2\nu + 2\mu^2\right) + \\ &\quad E[x_{t-1}x_t]\left(-2ww' + 4w\mu\right) + \qquad (17) \\ &\quad E[x_{t-1}^2]w^2 + 4p^2\left(\mu - \nu - \mu^2\right) + \\ &\quad \nu y^2 + 2\mu^2 y\hat{y} + \nu\hat{y}^2 \\ E[x_{t+1}x_t] &= E[x_t^2](w' - c) - wE[x_t x_{t-1}] + p^2 c \quad (18) \end{aligned}$$

We know that if (c, w) are additionally within the convergence region for the system, shown in Figure 3, then also $E[x_t^2]$ and $E[x_{t-1}x_t]$ will tend to a fixed-point. Let us find such fixedpoints. To do so we will assume we are at those fixed points, which we call p_{x^2} and p_{xx}, respectively. We substitute these into Equations (17) and (18) to obtain

$$\begin{aligned} p_{x^2} &= p_{x^2}\left(w'^2 - 4\mu w' + 2\nu + 2\mu^2\right) + \\ &\quad p_{xx}\left(-2ww' + 4w\mu\right) + \qquad (19) \\ &\quad p_{x^2}w^2 + 4p^2\left(\mu - \nu - \mu^2\right) + \nu y^2 + 2\mu^2 y\hat{y} + \nu\hat{y}^2 \\ p_{xx} &= p_{x^2}(w' - c) - wp_{xx} + p^2 c \qquad (20) \end{aligned}$$

The second equation allows us to compute

$$p_{xx} = p_{x^2}\left(1 - \frac{c}{w'}\right) + p^2\frac{c}{w'} \qquad (21)$$

Substitution of this in the first equation in (17) gives the fixed-points shown in Figure 4.

In order for a particle to converge, i.e., $\lim_{t\to\infty} x_t = p$, it is not enough to have $\lim_{t\to\infty} E[x_t] = p$: we must also have $\lim_{t\to\infty} StdDev[x_t] = 0$. This in turns requires $\lim_{t\to\infty} E[x_t^2] = p^2$. That is, we require $p_{x^2} = p^2$. To see when this can be the case, let us analyse Equation (22) in more detail.

For conciseness let us define

$$\begin{aligned} \Delta &= 1 - \Big[\left(w'^2 - 4\mu w' + 2\nu + 2\mu^2\right) \qquad (24) \\ &\quad + \left(1 - \frac{c}{w'}\right)2w\left(2\mu - w'\right) + w^2\Big]. \end{aligned}$$

Then, with little algebra one can see that

$$\begin{aligned} p_{x^2} &= \left(\frac{\Delta - 2(\mu^2 - \nu)}{\Delta}\right)p^2 \qquad (25) \\ &\quad + \left(\frac{4y(\mu^2 - \nu)}{\Delta}\right)p + \left(\frac{2y^2(\nu - \mu^2)}{\Delta}\right) \end{aligned}$$

where we used the substitution $\hat{y} = 2p - y$.

So, in general $p_{x^2} \neq p^2$ except if $y = p$, i.e., $\hat{y} = y$. Then $p_{x^2} = p^2$. So, except for the best particle in the swarm, the standard deviation of the sampling distribution, $StdDev[x_t]$, does not converge to 0, but to

$$\begin{aligned} p_{sd} & \\ &= \sqrt{\left(\frac{-2(\mu^2 - \nu)}{\Delta}\right)p^2 + \left(\frac{4y(\mu^2 - \nu)}{\Delta}\right)p + \left(\frac{2y^2(\nu - \mu^2)}{\Delta}\right)} \\ &= \sqrt{2\frac{(\nu - \mu^2)}{\Delta}\left(p^2 - 2yp + y^2\right)} \\ &= \sqrt{2\frac{(\nu - \mu^2)}{\Delta}(p - y)^2} \\ &= \sqrt{2\frac{(\nu - \mu^2)}{\Delta}} \cdot |p - y| \end{aligned}$$

$$p_{x^2} = \frac{4p^2\left(\mu-\nu-\mu^2\right)+\nu y^2+2\mu^2 y\hat{y}+\nu\hat{y}^2+p^2\frac{c}{w'}2w\left(2\mu-w'\right)}{1-\left[\left(w'^2-4\mu w'+2\nu+2\mu^2\right)+\left(1-\frac{c}{w'}\right)2w\left(2\mu-w'\right)+w^2\right]} \tag{22}$$

$$p_{xx} = \frac{4p^2\left(\mu-\nu-\mu^2\right)+\nu y^2+2\mu^2 y\hat{y}+\nu\hat{y}^2+p^2\frac{c}{w'}2w\left(2\mu-w'\right)}{1-\left[\left(w'^2-4\mu w'+2\nu+2\mu^2\right)+\left(1-\frac{c}{w'}\right)2w\left(2\mu-w'\right)+w^2\right]}\left(1-\frac{c}{w'}\right)+p^2\frac{c}{w'} \tag{23}$$

Figure 4: Fixed-points for $E[x_t^2]$ and $E[x_t x_{t-1}]$.

which we can finally rewrite as

$$p_{sd} = \sqrt{2\frac{(\nu-\mu^2)}{\Delta}}\cdot\left|\frac{\hat{y}-y}{2}\right| \tag{26}$$

Hence the search continues unless $y=\hat{y}$.

It is interesting to note that the observation that led to the definition of the bare-bones PSO [18] that the standard deviation of the search distribution is proportional to $\left|\frac{\hat{y}-y}{2}\right|$ was fundamentally correct. There is, however, a multiplicative factor, $\sqrt{2(\nu-\mu^2)/\Delta}$, in Equation (26) which depends on the parameters c and w and that was not previously detected. This factor may explain part of the differences in performance observed when comparing the bare-bones PSO and the classical algorithm.

4. NUMERICAL INTEGRATION

In this section we report the results of numerical integration of the dynamic equations for $E[x_t]$, $E[x_t^2]$ and $E[x_t x_{t-1}]$ (Equation (15)).

We start (Figure 5) by considering the case $c = 1.49618$ and $w = 0.7298$ which corresponds to the parameter values recommended in [5] for the PSO with constriction. Note how, while $E[x_t]$ converges to $p = 3$ within 30 generations, $StdDev[x_t]$ never converges to zero, settling onto a value of just over 2.0 within about 70 generations. The picture is very different, however, if $y = \hat{y}$, as shown in Figure 6. In this case, $E[x_t^2]$ and $E[x_t x_{t-1}]$ converge to p^2. As a result, $StdDev[x_t]$ decreases to zero. The decrease is exponential, corroborating Blackwell's analysis of how the spatial extent of a particle swarm varies over time [19] (see Section 1).

Examples of configurations where the mean converges to its fixed-point while $StdDev[x_t]$ does not converge to 0 are shown in Figure 8. Note that this is not necessarily an undesirable behaviour. In some situations having a sampling distribution that progressively widens if improvements cannot be found might be exactly what one needs. What is important is to be able to control whether or not there is growth of $StdDev[x_t]$ and a what rate. This is exactly what our model allows one to do.

By an appropriate setting of parameters we can even achieve a self-limiting growth in $StdDev[x_t]$, and, furthermore, we can fix its asymptote by design. A way to achieve this is to note that if $c = w' = 1 + w$, the fixed-point for $E[x_t x_{t-1}]$ in Equation (21) simplifies to $p_{xx} = p^2$. Then we have

$$\begin{aligned} p_{x^2} = {} & p_{x^2}\left(w'^2-4\mu w'+2\nu+2\mu^2\right)+ \\ & p^2\left(-2ww'+4w\mu\right)+ \\ & p_{x^2}w^2+4p^2\left(\mu-\nu-\mu^2\right)+\nu y^2+2\mu^2 y\hat{y}+\nu\hat{y}^2 \end{aligned} \tag{27}$$

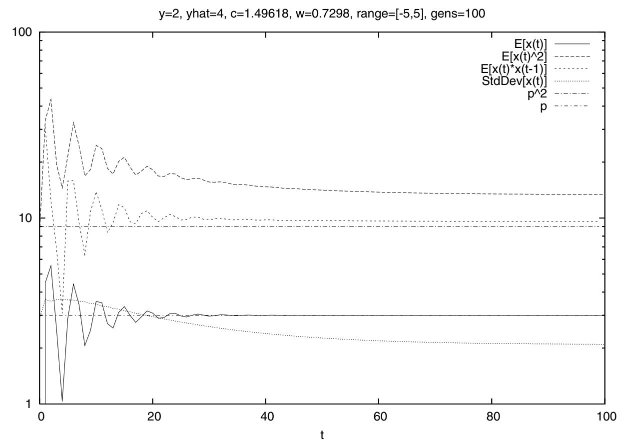

Figure 5: Numerical integration of the system of difference equations in Equation (15) for $c = 1.49618$, $w = 0.7298$, $y = 2$, $\hat{y} = 4$ and $\Omega = 5$.

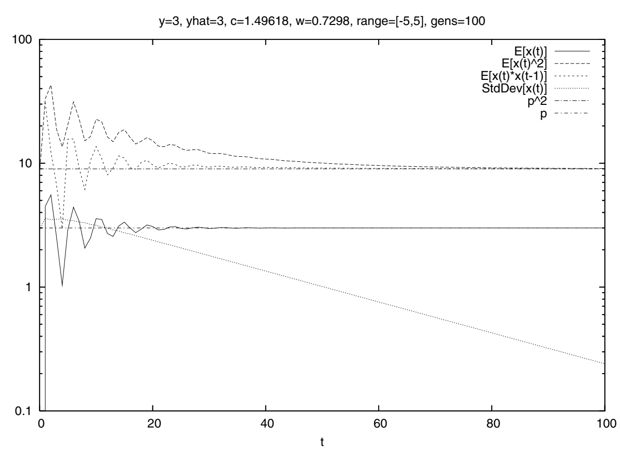

Figure 6: Numerical integration of the system of difference equations in Equation (15) for $c = 1.49618$, $w = 0.7298$, $y = \hat{y} = 3$ and $\Omega = 5$.

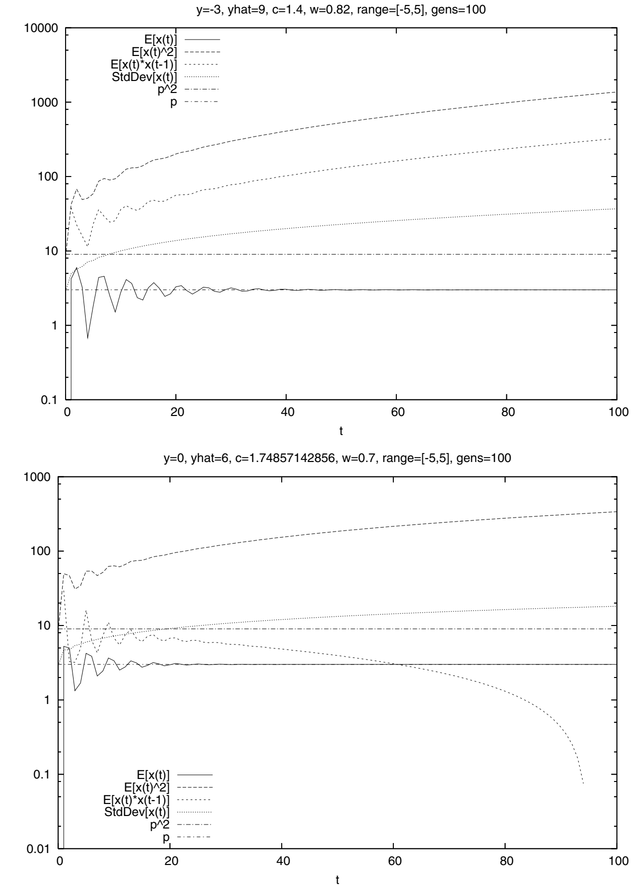

Figure 8: Numerical integration of the system of difference equations in Equation (15) for values of c and w where $E[x_t]$ is convergent, but $StdDev[x_t] \not\to 0$.

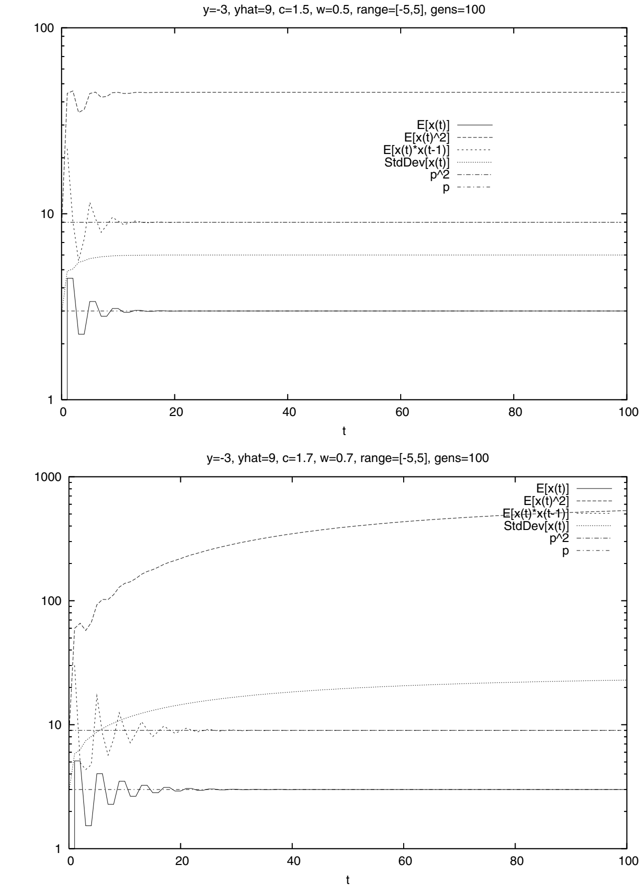

Figure 9: Numerical integration of the system of difference equations in Equation (15) for values of c and w where $E[x_t]$ is convergent to p, $E[x_t x_{t-1}]$ is convergent to p^2 and $E[x_t^2]$ is convergent to the value given in Equation (28).

which can be solved for p_{x^2}, obtaining, after simplification, the fixed-point in Figure 7. With this in hand one can then compute $p_{sd} = \sqrt{p_{x^2} - p^2}$. For example, for $w = 0.5$, $y = -3$ and $\hat{y} = 9$, we obtain $p_{x^2} = 45$ and $p_{sd} = \sqrt{45 - 9} = 6$, while for $w = 0.7$ one obtains $p_{x^2} = 621$ and $p_{sd} = \sqrt{621 - 9} = 24.739$. As one can see in Figure 9, there are indeed the asymptotes to which the system converges.

5. DISCUSSION AND CONCLUSIONS

Several theoretical analyses of the dynamics of particle swarms have been offered in the literature over the last decade. These have been very illuminating. However, virtually all have relied on substantial simplifications, and in particular on the assumption that the particles are deterministic. Naturally, this assumption makes it impossible to derive an exact characterisation of the sampling distribution of the PSO. This distribution has therefore remained the "holy grail" of PSO research for almost a decade.

By using of surprisingly simple techniques, in this paper we have been able to exactly determine perhaps the most important characteristic of a PSO's sampling distribution, its variance, and we have been able to explain how it changes change over any number of generations. The only assumption we made is stagnation, so our characterisation is valid for as long as a particle searches for a better personal best.

We applied the analysis to the PSO with inertia weight, but, as we explained in Section 1, the analysis is also valid for the PSO with constriction via a simple parameter mapping.

Knowing the dynamics of the variance of the PSO's sampling distribution and being able to control it, as, for example, we illustrated in Section 4, is very important because it allows one to understand the search behaviour of the PSO and adapt it to a problem at hand.

The dynamics of the variance of the PSO's sampling distribution is also important from a theoretical standpoint. In order for a particle to converge, it is not enough to require $\lim_{t\to\infty} E[x_t] = p$: we must also have $\lim_{t\to\infty} StdDev[x_t] = 0$. In the absence of accurate information on $StdDev[x_t]$, previous research has effectively assumed that $\lim_{t\to\infty} E[x_t] = p$ would eventually drive $StdDev[x_t]$ to zero. This assumption has, for example, been used in the proof provided in [6] and [2] that the PSO is not guaranteed to be an optimiser. However, as we have shown in this work, $\lim_{t\to\infty} StdDev[x_t] = 0$ only if $y = \hat{y}$, and so whether or not the PSO is an optimiser is still effectively a conjecture.

How could our results help obtain a formal proof of convergence? The stagnation assumption essentially removes

$$p_{x^2} = \frac{(w+1)^2(2p-y)^2/3 + 0.5y(2p-y)(w+1)^2 + y^2(w+1)^2/3 + 2(w - \frac{7}{6}(w+1)^2+1)p^2}{-\frac{1}{6}(w+1)^2 - w^2 + 1} \quad (28)$$

Figure 7: Fixed-point for $E[x^2]$ when $c = 1 + w$.

the dependence on the details of the fitness function. Our results can be used to prove convergence when stagnation has occurred. So, a proof of convergence for the PSO would require finding under which conditions and for what fitness functions the system stagnates. We will pursue this line of attack in future research.

Although we have applied our analysis only to first and second order statistics of the sampling distribution, nothing prevents one from constructing higher order statistics. In future research we want to study these. This would provide us with a deeper theoretical characterisation of the PSO. Because of the complexity of the calculations involved, this may require mechanisation via a symbolic algebra system.

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